On the Inefficiency of Pay-As-You-Go Pension Systems in Stochastic Economies with Assets

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Discussion Paper No. 565

June 2007

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Abstract

This paper studies the impact of variations of a pay-as-you-go pension system on welfare in a stochastic overlapping generations model where compulsory public retirement savings coexist with private savings in assets and bonds. It is shown that, for a stationary population, any reduction of contribution rates once and for all leads to a long-run welfare improvement of consumers. Moreover, a gradual reduction of contribution rates induces a smooth transition towards a purely privately generated retirement system keeping the welfare losses of current retirees sufficiently small. An unfavorable demographic change, modelled as a transitory phenomenon toward increased ageing of the population, is shown to induce significant welfare losses. The paper discusses several possible adjustment policies. It is shown that in general an adjustment of contribution rates cannot alleviate but may even amplify the welfare loss. The most promising political measure is shown to be a temporary increase in the retirement age.

Keywords: Pension System, Pay-As-You-Go, Welfare, Asset Market, OLG

JEL Classification: D9, E44, H55

First version: Sept. 2006, this version: July 4, 2007.

*Acknowledgment. This paper extends parts of the Dissertation of M. Hillebrand (2006). We are indebted to Thorsten Pampel, Olaf Schmitz, and Jan Wenzelburger for many discussions.

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Introduction

Demographic scenarios for the next fifty years predict significant structural changes in the populations of almost all industrialized countries. These demographic changes put increasing pressure on existing pension systems as they increase the number of beneficiaries relative to the number of contributors to the systems. While the magnitude of change varies depending on the underlying population scenario, it is generally accepted that without substantial reforms the existing pay-as-you-go pension systems will be unable to provide the standard of living they offered in the past.

Against this background, there has been an intensive debate about the design and efficiency of pension systems and numerous reform proposals have been discussed. For Germany this has resulted in a fundamental pension reform at the beginning of this century. The essential idea of this reform is to supplement the existing public pension system by an increased share of private savings for retirement. While this measure potentially fosters the accumulation of capital and may thus attenuate the loss in workforce induced by the demographic change, opponents of the reform have argued that savings are exposed to capital market risk. As a consequence, the larger share of private savings would necessarily increase the risk to which pension incomes are subjected due to the unpredictability of asset markets, in particular of stock markets. The latter argument suggests that any theoretical study of pension systems should pay adequate respect to the role of risk and uncertainty. Conceptually, this calls for a macroeconomic model which incorporates the issue of demographic change and the random influence of a stochastic asset market. If, in addition, the goal is a comparison of alternative pension reforms in terms of their optimality, normative concepts for stochastic models are required to measure the impact on consumer welfare.

The literature on pension systems mostly confines itself to a deterministic framework. Examples may be found in Breyer (1989), Breyer & Straub (1993), Brunner (1994) or Homburg (1990). Here the traditional concept of Pareto optimality offers a widely-accepted tool to assess and compare the welfare effects of alternative pension reforms. Extensions of this concept to a stochastic setting may be found, e.g., in Demange & Laroque (1999), Demange (2002), Gottardi & Kübler (2006) and Krüger & Kübler (2006). These include the notions of *interim Pareto efficiency* and *ex-ante optimality* which may be used to analyze and compare the efficiency of pension systems in a random environment. Despite their appeal the formulation and application of these concepts is closely tied to the particular stochastic setting adopted in these models, where the underlying probability space is discrete and the randomness can be represented by a so-called date-event tree. While this permits the study of welfare issues within the setting of incomplete markets (see Magill & Quinzii (1998)), the framework is essentially static and does not allow comparisons of alternative outcomes of the model on a time series level.

Welfare comparisons in deterministic models of economic growth are sometimes confined to stationary states, leading to the so-called golden rule criterion, see, e.g., Phelps

(1961). It provides a possibility to evaluate the long-run outcome of the model depending on parameters of interest. This concept has been generalized and applied to a stochastic version of the Solow growth model (see Schenk-Hoppé 1999), showing existence of an optimal savings propensity inducing maximal mean consumption under stationarity capital accumulation. It seems that such an approach provides tools and methods from random dynamical systems theory which allows a comparison of alternative stationary solutions under different settings as well as an evaluation of empirically observed time series. The welfare criteria used in this paper are based on these methods.

The present paper provides an explicit dynamic framework to analyze the interaction between the pension system, the population structure, and real and financial markets in the presence of arbitrary random shocks to the system. Given its form as a random dynamical system allows a comparative analysis of stationary solutions of alternative pension scenarios under arbitrary random perturbations. The framework also permits the study of transition phases under demographic change of the population and of adjustments of the pension parameters, allowing to quantify utility gains and losses of generations implied by alternative adjustment policies and demographic scenarios.

The paper is organized as follows. Section 1 reviews the underlying theoretical model and its basic assumptions. Section 2 presents the dynamic structure of the model within the theory of random dynamical system. Sections 3 and 4 formulate a long-run welfare criterion and study the role of a pension system for the case with a stationary population. Section 5 studies the welfare effects of a transition from a pay-as-you-go towards a fully private system with an asset market. The impact of demographic change on consumer welfare is studied in Section 6, followed by a discussion of various adjustments of the pension systems in Sections 7 and 8. Section 9 draws some conclusions. The appendix collects basic concepts from random dynamical systems theory and mathematical proofs.

1 The model

Consider an economy with overlapping generations of homogeneous consumers who live for $J+1 \geq 2$ time periods implying that in every period $t \in \mathbb{N}_0$ there are $J+1$ different generations in the market. Each generation is identified by the index $j \in \{0, 1, \dots, J\}$ describing the remaining lifetime of each consumer in this generation. Let $N_t^{(j)} > 0$ denote the number of consumers in generation j at time t and define the population vector $N_t = (N_t^{(j)})_{j=0}^{J}$. Each consumer in generation $j \in \{j_L, \dots, J\}$ supplies $\bar{L}^{(j)} > 0$ units of labor inelastically to the labor market where the threshold $j_L > 0$ denotes the number of retired generations. Aggregate labor supply at time t is thus given by

$$L_t^S := \sum_{j=j_L}^{J} \bar{L}^{(j)} N_t^{(j)}. \tag{1}$$

There is a single consumption good in the economy which serves as numeraire. Let $\omega_t > 0$ denote the gross real wage per unit of labor at time t out of which a fraction

$\tau_t \in [0,1[$ has to be contributed to the public pension system. Then each working consumer earns net labor income

$$e_t^{(j)} = (1-\tau_t)\,\omega_t\,\bar{L}^{(j)} > 0, \qquad j = j_L, \ldots, J. \tag{2}$$

at time t. The pension system is a pure pay-as-you-go system where total contributions are divided up equally among current retirees in each period. The non-capital income of each retired consumer at time t is thus

$$e_t^{(j)} = e_t^R := \tau_t \frac{\omega_t\, L_t^S}{\sum_{j=0}^{j_L-1} N_t^{(j)}} \geq 0, \qquad j = 0, \ldots, j_L - 1. \tag{3}$$

To transfer income between different time periods there are two savings possibilities available to each consumer. The first one is a one-period bond which is traded at a price of unity at time t and pays a non-random return $R_t > 0$ in the following period $t+1$. Since R_t is determined at time t, the bond provides a riskless investment possibility between any two consecutive periods. The second investment possibility are stocks of a firm (or shares) which are traded at price $p_t > 0$ in each period t and which pay a random-dividend $d_t \geq 0$ prior to trading. Dividends are generated endogenously from the production activities of the firm. The total number of shares in the market is constant and denoted as $\bar{x} > 0$. Bonds may be sold short without bound while short selling of shares is not possible. Thus, the bond provides a possibility for consumers to obtain also credit.

A typical consumer belonging to generation $j \in \{0, 1, \ldots, J\}$ solves an expected utility maximization problem to determine his current consumption and investment in bonds and stocks. His wealth position at time t consists of his capital income corresponding to the return on his previous asset investment and his non-capital income determined by equations (2) and (3), respectively. Let $(y_t^{(j)}, x_t^{(j)}) \in \mathbb{R} \times \mathbb{R}_+$ denote the portfolio held by a consumer belonging to generation $j \in \{1, \ldots, J\}$ after trading in period t consisting of bond investment $y_t^{(j)}$ and share holdings $x_t^{(j)}$. Since the capital income of young consumers is zero, the wealth of a consumer in generation j at time t is

$$w_t^{(j)} := \begin{cases} e_t^{(j)}, & j = J \\ e_t^{(j)} + R_{t-1}\, y_{t-1}^{(j+1)} + x_{t-1}^{(j+1)}\,(p_t + d_t) & j = 0, 1, \ldots, J-1. \end{cases} \tag{4}$$

At time t the consumer holds subjective expectations $\hat{e}_t^{(j)} := (\hat{e}_{t,t+n}^{(j)})_{n=1}^{j} \in \mathbb{R}_+^j$ for his future non-capital income with $\hat{e}_{t,t+n}^{(j)} \geq 0$ denoting the forecast for his non-capital income in period $t+n$. Likewise he holds expectations $\hat{R}_t^{(j)} := (\hat{R}_{t,t+n})_{n=1}^{j-1} \in \mathbb{R}_{++}^{j-1}$ for future bond returns where $\hat{R}_{t,t+n}$ is the point forecast for the bond return R_{t+n} between future periods n and $n+1$, $n \in \{1, \ldots, j-1\}$. Future asset prices and dividends are treated as random variables in the decision. The consumer's expectations for future asset prices and dividends within his planning horizon take the form of a subjective joint probability distribution $\hat{\nu}_t$ of the random variables $(p_{t+n}, d_{t+n})_{n=1}^{j}$. The consumer's decision problem at time t involves the choice of a consumption-investment strategy that specifies a consumption and investment decision for period t and mutually consistent plans for all future periods $t+1, \ldots, t+j$ within his remaining lifetime. His preferences over alternative consumption streams are represented by the logarithmic utility function

$$(c_t, c_{t+1}, \ldots, c_{t+j}) \longmapsto \sum_{n=0}^{j} \beta^n \ln(c_{t+n}), \quad \beta > 0. \tag{5}$$

The consumer's objective is to maximize the expected utility of consumption within his remaining lifetime. Let $q_{t+1} := p_{t+1} + d_{t+1}$ denote next period's cum-dividend price with induced distribution ν_t derived form $\hat{\nu}_t$. It is well known that the logarithmic specification (5) induces myopic investment behavior such that the demand at time t is completely determined by the measure ν_t. In the sequel we assume that ν_t is taken from a fixed class of elliptically symmetric distributions parameterized in $(\mu_t, \sigma_t) \in \mathbb{R}^2_{++}$ with compact support $\bar{\mathbb{Q}}_t = \bar{\mathbb{Q}}(\mu_t, \sigma_t)$. This class is generated by some random variable ε with symmetric distribution ν_ε supported on $[-\bar{\varepsilon}, \bar{\varepsilon}]$ where $\bar{\varepsilon} > 0$ is a given constant[1] . The perceived distribution $\nu_t = \nu_{\mu_t,\sigma_t}$ of the random variable q_{t+1} is given as the image measure of ν_ε under the affine map $\varepsilon \longmapsto \mu_t + \sigma_t \varepsilon$. The pair (μ_t, σ_t) defines the mean and dispersion of the distribution and will be called the consumers' beliefs about q_{t+1}. To ensure that $\bar{\mathbb{Q}}(\mu_t, \sigma_t) \subset \mathbb{R}_{++}$ we restrict attention to the set $\mathbb{B} = \{(\mu, \sigma) \in \mathbb{R}^2_{++} | \mu > \sigma\, \bar{\varepsilon}\}$ of feasible beliefs.

Let $\hat{\epsilon}_t^{(j)} := \hat{e}_{t,t+1}^{(j)} + \hat{e}_{t,t+2}^{(j)}/\hat{R}_{t,t+1} + \ldots + \hat{e}_{t,t+j}^{(j)}/(\hat{R}_{t,t+1} \cdots \hat{R}_{t,t+j-1})$ denote the discounted non-capital income stream derived from the expectations $\hat{e}_t^{(j)}$ and $\hat{R}_t^{(j)}$. Given the current bond return $R > 0$, the buying price $p > 0$, and wealth $w > -\hat{\epsilon}_t^{(j)}$ determined by (4) it can be shown that optimal consumption and investment in stocks and bonds at time t are determined by the following demand functions

$$\begin{aligned}
\varphi_c^{(j)}(R, p, w; \mu_t, \sigma_t, \hat{e}_t^{(j)}, \hat{R}_t^{(j)}) &= \bar{c}^{(j)}(w + \hat{\epsilon}_t^{(j)}/R) \\
\varphi_x^{(j)}(R, p, w; \mu_t, \sigma_t, \hat{e}_t^{(j)}, \hat{R}_t^{(j)}) &= (1 - \bar{c}^{(j)})(w + \hat{\epsilon}_t^{(j)}/R)\, \theta(R\,p; \mu_t, \sigma_t) \\
\varphi_y^{(j)}(R, p, w; \mu_t, \sigma_t, \hat{e}_t^{(j)}, \hat{R}_t^{(j)}) &= (1 - \bar{c}^{(j)})(w + \hat{\epsilon}_t^{(j)}/R)(1 - p\, \theta(R\,p; \mu_t, \sigma_t)) - \hat{\epsilon}_t^{(j)}/R.
\end{aligned} \tag{6}$$

Here $\bar{c}^{(j)} := \left[1 + \beta + \ldots + \beta^j\right]^{-1}$ and the share of (lifetime) income invested in shares is determined by the map $\theta : \mathbb{R}_{++} \times \mathbb{B} \longrightarrow [0, 1]$,

$$\theta(\pi; \mu_t, \sigma_t) := \arg\max_{\vartheta} \left\{ \int_{[-\bar{\varepsilon}, \bar{\varepsilon}]} \ln(\pi + \vartheta\, (\mu_t + \sigma_t \varepsilon - \pi))\; \nu_\varepsilon(d\,\varepsilon) \;\; \middle| \; \vartheta \in [0, 1] \right\}. \tag{7}$$

Since $c^{(0)} = 1$ the functions (6) also describe the demand behavior of the old generation $j = 0$ whose members consume only and do not invest.

There is a single firm which produces the consumption good in period t using its capital stock $K_t > 0$ and labor $L_t \geq 0$. In addition production at time t is subject to a random shock term η_t taking values in the compact interval $[0, \eta_{max}]$. The firm's production technology is described by the production function $F : \mathbb{R}^2_+ \times [0, \eta_{max}] \longrightarrow \mathbb{R}_+$

$$F(K_t, L_t, \eta_t) := \kappa K_t^{1-\alpha} L_t^{\alpha} + \eta_t. \tag{8}$$

[1] In the simulations we take ν_ε to be a standard normal distribution truncated to the interval $[-\bar{\varepsilon}, \bar{\varepsilon}]$.

In each period t, the capital stock $K_t > 0$ is given and the firm decides about labor demand $L_t \geq 0$ and investment $I_t \geq 0$. The latter determines the next capital stock as

$$K_{t+1} = I_t + (1-\delta)K_t \tag{9}$$

where $\delta \in]0,1]$ denotes the constant rate of depreciation. To extend its capital stock the firm can transform the consumption good into capital. As in Abel (2003), suppose that given the current capital stock $K_t > 0$, the amount of consumption goods needed to produce $I_t \geq 0$ units of new capital is determined by the adjustment cost function $G : \mathbb{R}_+ \times \mathbb{R}_{++} \longrightarrow \mathbb{R}_+$, $G(I, K) := K\, g(I/K)$. The properties of the function G are mainly determined by the function g which depends on the investment ratio $i := I/K$. The function g is assumed to be of the form

$$g(i) = \begin{cases} 0 & i = 0 \\ \gamma_0 \exp\{\gamma_1 i\} & i > 0 \end{cases}, \quad \gamma_0 > 0, \gamma_1 > \frac{1}{\delta}. \tag{10}$$

The firm's investment I_t at time t is exclusively financed by issuing one period bonds $B_t \geq 0$ inducing the obligation to pay $R_t B_t$ units of output/consumption good at time $t+1$. Recalling that the bond price is normalized to unity, one finds that investment and bond supply at time t are related by

$$B_t = G(I_t, K_t) = K_t\, g(I_t/K_t). \tag{11}$$

After paying for labor and the bond debt incurred in the previous period, the firm distributes all excess output as a dividend payment to its shareholders. Letting as before $\bar{x} > 0$ denote the total number of shares in the market, the dividend payment (per share) at time t is given by

$$d_t \;=\; \frac{F(L_t, K_t, \eta_t) - \omega_t L_t - R_{t-1} B_{t-1}}{\bar{x}}. \tag{12}$$

The firm seeks to act in favor of its shareholders by choosing labor input L_t to maximize current dividends and investment I_t to maximize next period's expected dividend payment. Assume that the firm holds expectations $\hat{\omega}_{t,t+1}$ for next period's real wage. Given the objective, let

$$L(\omega; K_t) := \left(\frac{\alpha\kappa}{\omega}\right)^{\frac{1}{1-\alpha}} K_t \tag{13}$$

denote the firm's cost minimizing labor demand. Similarly, given expectations $\hat{\omega}_{t,t+1} > 0$ optimal investment is determined by the function

$$I(R; \hat{\omega}_{t,t+1}, K_t) := \begin{cases} \frac{1}{\gamma_1} \ln\left(\frac{(1-\alpha)\kappa}{\gamma_0\, \gamma_1\, R} \left(\frac{\alpha\kappa}{\hat{\omega}_{t,t+1}}\right)^{\frac{\alpha}{1-\alpha}} \right) K_t & 0 < R < \bar{R}(\hat{\omega}_{t,t+1}) \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

where $\bar{R}(\hat{\omega}_{t,t+1}) := \frac{(1-\alpha)\kappa}{e\,\gamma_0\,\gamma_1} \left(\frac{\alpha\kappa}{\hat{\omega}_{t,t+1}}\right)^{\frac{\alpha}{1-\alpha}}$. In the sequel we will restrict attention to interior investment solutions by assuming that $R_t < \bar{R}(\hat{\omega}_{t,t+1})$ for all times t. In this case, it follows from (11) that the firm's bond-supply is determined by the function

$$B(R; \hat{\omega}_{t,t+1}, K_t) \;:=\; \frac{(1-\alpha)\kappa}{\gamma_1\, R} \left(\frac{\alpha\kappa}{\hat{\omega}_{t,t+1}}\right)^{\frac{\alpha}{1-\alpha}} K_t. \tag{15}$$

The demand behavior by consumers and by the firm determines market clearing prices on real and financial markets in every period $t \in \mathbb{N}_0$ endogenously. Given the population vector N_t, let labor supply $L_t^S > 0$ be defined as in (1). Given the capital stock $K_t > 0$, labor demand is given by equation (13). Assuming full employment the real wage ω_t is determined such that $L(\omega_t; K_t) = L_t^S$. Utilizing (13) this implies the existence of a non-random map $\mathcal{W} : \mathbb{R}_{++}^2 \longrightarrow \mathbb{R}_{++}$ which determines the real wage as

$$\omega_t = \mathcal{W}(K_t, L_t^S) := \alpha\kappa \left(K_t/L_t^S\right)^{1-\alpha}. \tag{16}$$

For a given contribution rate τ_t the equilibrium real wage (16) implies a non-capital income distribution $e_t = (e_t^{(j)})_{j=0}^J$ defined by equations (2) and (3). Likewise, given the production shock η_t at time t, the dividend payment d_t is determined by equation (12). Based on these observations consumers in each generation $j \in \{1, \ldots, J\}$ form their expectations $\hat{e}_t^{(j)} = (\hat{e}_{t,t+n}^{(j)})_{n=1}^j$ and $\hat{R}_t^{(j)} = (\hat{R}_{t,t+n})_{n=1}^j$ for future non-capital income and future bond returns and determine their beliefs (μ_t, σ_t) for asset prices. Likewise the firm determines its point forecast $\omega_{t,t+1}$ for next period's real wage. Given the lists of expectations $\hat{e}_t := (\hat{e}_t^{(j)})_{j=1}^J$ and $\hat{R}_t := (\hat{R}_t^{(j)})_{j=2}^J$ as well as the values e_t and d_t together with the previous asset allocation $z_{t-1} := (y_{t-1}^{(j)}, x_{t-1}^{(j)})_{j=1}^J$ among consumers, the bond return R_t and the share price p_t are determined simultaneously such that market clearing on the bond market and the stock market obtains. Given the particular functional forms of the asset demand functions (6) and (15) it is shown in Hillebrand (2007) that, letting $\hat{m}_t := \frac{1}{\bar{x}}\left[\sum_{j=1}^J B(1; \hat{\omega}_{t,t+1}, K_t) + N_t^{(j)}\hat{\epsilon}_t^{(j)}\right]$ there exists a mapping $\pi : \mathbb{B} \times \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$

$$\pi(\mu_t, \sigma_t, \hat{m}_t) := \frac{\vartheta(\mu_t, \sigma_t, \hat{m}_t)\, \hat{m}_t}{1 - \vartheta(\mu_t, \sigma_t, \hat{m}_t)} \quad \text{where} \quad \vartheta(\mu_t, \sigma_t, \hat{m}_t) := \int_{[-\bar{\varepsilon},\bar{\varepsilon}]} \frac{\mu_t + \sigma_t\varepsilon}{\mu_t + \sigma_t\varepsilon + \hat{m}_t}\, \nu_\varepsilon(d\,\varepsilon)$$

such that equilibrium prices are determined as

$$\begin{aligned} R_t &= \mathcal{R}(N_t, K_t, e_t, d_t, \mu_t, \sigma_t, \hat{e}_t, \hat{R}_t, \hat{\omega}_{t,t+1}, z_{t-1}, R_{t-1}) \qquad (17)\\ &:= \frac{B(1; \hat{\omega}_{t,t+1}, K_t) + \sum_{j=1}^J N_t^{(j)}\bar{c}^{(j)}\hat{\epsilon}_t^{(j)} + \pi(\mu_t, \sigma_t, \hat{m}_t)\sum_{j=0}^{J-1} N_t^{(j)}\bar{c}^{(j)}x_{t-1}^{(j+1)}}{\sum_{j=1}^J N_t^{(j)}(1-\bar{c}^{(j)})e_t^{(j)} + \sum_{j=1}^{J-1} N_t^{(j)}(1-\bar{c}^{(j)})\left[R_{t-1}y_{t-1}^{(j+1)} + d_t\, x_{t-1}^{(j+1)}\right]} \end{aligned}$$

$$\begin{aligned} p_t &= \mathcal{P}(N_t, K_t, e_t, d_t, \mu_t, \sigma_t, \hat{e}_t, \hat{R}_t, \hat{\omega}_{t,t+1}, z_{t-1}, R_{t-1}) \qquad (18)\\ &:= \frac{\pi(\mu_t, \sigma_t, \hat{m}_t)}{\mathcal{R}(N_t, K_t, e_t, d_t, z_{t-1}, R_{t-1}, \mu_t, \sigma_t, \hat{e}_t, \hat{R}_t, \hat{\omega}_{t,t+1})}. \end{aligned}$$

Utilizing (4) and (6) the equilibrium prices (17) and (18) determine a new asset allocation $z_t := (y_t^{(j)}, x_t^{(j)})_{j=1}^J$ and define the consumption $c_t := (c_t^{(j)})_{j=0}^J$ of the consumers in each generation. The firm uses the consumption goods collected from its bond sales to form $I_t > 0$ units of new capital which determines the capital stock K_{t+1} according to (9).

2 Dynamics

To complete the description of the model the evolution of the population as well as the expectations formation is made explicit. Since life expectancy is deterministic and constant one has for each $t \in \mathbb{N}_0$

$$N_t^{(j)} = N_{t-1}^{(j+1)}, \quad j = 0, 1, \ldots, J-1. \tag{19}$$

The number $N_t^{(J)}$ of young consumers born at time t is determined from the previous population N_{t-1} according to the map $\mathcal{N} : \mathbb{R}_+^{J+1} \longrightarrow \mathbb{R}_+$

$$\mathcal{N}(N_{t-1}) := \sum_{j=0}^{J} N_{t-1}^{(j)} n^{(j)} \Big(1 + \exp\Big\{-n_2 \sum_{i=0}^{J} N_{t-1}^{(i)}\Big\}\Big), \; n^{(j)} \geq 0, \; j = 0, \ldots, J, \; n_2 > 0. \tag{20}$$

To describe the forecasting behavior of consumers and the firm, recall that the information set upon which expectations at time t are based contains the current real wage ω_t and the non-capital income distribution e_t but not the current price p_t and the bond return R_t. Suppose that the firm's real wage prediction satisfies

$$\hat{\omega}_{t,t+1} = \omega_t \tag{21}$$

for all times $t \in \mathbb{N}_0$. Consumers in generation $j \in \{1, \ldots, J\}$ derive their non-capital income expectation $\hat{e}_{t,t+n}^{(j)}$ for period $t+n$ from the current income of generation $j-n$ (corresponding to their age at $t+n$) such that for each $t \in \mathbb{N}_0$

$$\hat{e}_{t,t+n}^{(j)} = e_t^{(j-n)}, \quad n = 1, \ldots, j, \; j = 1, \ldots, J. \tag{22}$$

The prediction for future bond returns is assumed to be uniformly equal to the last observed bond return such that for each $t \in \mathbb{N}_0$

$$\hat{R}_{t,t+n} = R_{t-1}, \quad n = 1, \ldots, J-1. \tag{23}$$

Consumers' second moment beliefs about asset prices are constant ($\sigma_t \equiv \sigma$) while first moments μ_t are updated according to an adaptive error-correction principle such that

$$\mu_t = \mathcal{M}(\mu_{t-2}, q_{t-1}) := \mu_{t-2} + \varrho(q_{t-1} - \mu_{t-2}), \quad 0 \leq \varrho \leq 1. \tag{24}$$

Note that (24) includes the cases of naive expectations ($\varrho = 1 \Rightarrow \mu_t = q_{t-1}$) and of static expectations ($\varrho = 0 \Rightarrow \mu_t = \mu_{t-2}$). The following assumption specifies the stochastic nature of the noise process in (8).

Assumption 1
The process $\{\eta_t\}_t$ consists of i.i.d. random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The process is adapted to some filtration $\{\mathcal{F}_t\}_t$. Each η_t has a symmetric distribution ν_η supported on the set $[0, \eta_{max}]$ such that $\mathbb{E}_{\nu_\eta}[\eta_t] = \bar{\eta} = \eta_{max}/2$.

To formalize the dynamic evolution of the model, assume that the contribution rate to the pension system is constant such that $\tau_t \equiv \tau$. As before let $y_t = (y_t^{(j)})_{j=1}^J$ and $x_t = (x_t^{(j)})_{j=1}^J$ for each t and define the state vector

$$\xi_t := \left(N_t^\top, K_t, \mu_{t-1}, \mu_t, R_t, q_t, y_t^\top, x_t^\top\right)^\top. \tag{25}$$

We seek to obtain the model's evolution as a random difference equation of the form $\xi_t = \phi_\tau(\xi_{t-1}, \eta_t)$ for some suitable map ϕ_τ. In what follows the dynamics may be separated into the following three building blocks:

(i) *The dynamics of the population.* Combining (19) and (20) these are determined by the map $\hat{\mathcal{N}} : \mathbb{R}_{++}^{J+1} \longrightarrow \mathbb{R}_{++}^{J+1}$, $N_{t-1} \longmapsto \hat{\mathcal{N}}(N_{t-1}) = N_t$ defined by the conditions

$$\begin{cases} N_t^{(j)} &= N_{t-1}^{(j+1)}, \qquad j = 0, 1, \ldots, J-1 \\ N_t^{(J)} &= \mathcal{N}(N_{t-1}). \end{cases} \tag{26}$$

(ii) *The dynamics of capital.* Combining the law of motion (27) with the investment function (14), the wage law (16) and the forecasting rule (21) the capital stock evolves as

$$\begin{aligned} K_t &= \hat{\mathcal{K}}(N_{t-1}, K_{t-1}, R_{t-1}) := I(R_{t-1}; \mathcal{W}(K_{t-1}, L_{t-1}^S), K_{t-1}) + (1-\delta)K_{t-1} \\ &= \left[\frac{1}{\gamma_1} \ln\left(\frac{(1-\alpha)\kappa}{\gamma_0\gamma_1 R_{t-1}} \left(\frac{L_{t-1}^S}{K_{t-1}}\right)^\alpha\right) + 1 - \delta\right] K_{t-1}. \end{aligned} \tag{27}$$

(iii) *Financial dynamics.* These comprise the processes of mean beliefs, asset prices and asset allocations. Each of these may be treated separately such that the financial dynamics may be split into the following three blocks.
(a) *The process of mean forecasts.* Using (24) and taking proper account of the lag-structure the updating of mean forecasts are given by the linear map

$$(\mu_{t-2}, \mu_{t-1}, q_{t-1}) \longmapsto \hat{\mathcal{M}}(\mu_{t-2}, \mu_{t-1}, q_{t-1}) := (\mu_{t-1}, \mathcal{M}(\mu_{t-2}, q_{t-1})). \tag{28}$$

(b) *Dynamics of asset prices.* Given the fixed contribution rate τ the forecasting rule (22) together with equations (2), (3), and the wage law (16) imply that the non-capital income distribution e_t as well as consumers' expectations $\hat{e}_t$ for future non-capital income may be written as functions of N_t and K_t. By virtue of equations (1), (16) and (21) the same is true for the firm's real wage prediction. This together with (23) implies that the price laws (17) and (18) are of the following functional form

$$\begin{aligned} R_t &= \mathcal{R}_\tau(N_t, K_t, d_t, \mu_t, z_{t-1}, R_{t-1}) \\ p_t &= \mathcal{P}_\tau(N_t, K_t, d_t, \mu_t, z_{t-1}, R_{t-1}). \end{aligned} \tag{29}$$

Using (12) and (15) together with the wage law (16) the firm's dividend payment at time t takes the form

$$\begin{aligned} d_t &= \frac{F(L_t^S, K_t, \eta_t) - \mathcal{W}(K_t, L_t^S)L_t^S - B(1, \mathcal{W}(K_{t-1}, L_{t-1}^S), K_{t-1})}{\bar{x}} \\ &= \frac{(1-\alpha)\kappa}{\bar{x}}\left[(L_t^S)^\alpha K_t^{1-\alpha} - \frac{1}{\gamma_1}(L_{t-1}^S)^\alpha K_{t-1}^{1-\alpha}\right] + \frac{\eta_t}{\bar{x}}. \end{aligned} \tag{30}$$

Using (26) and (27) in (30) it is straightforward to express the dividend payment as $d_t = \hat{\mathcal{D}}(N_{t-1}, K_{t-1}, R_{t-1}) + \eta_t/\bar{x}$. Exploiting this together with (26), (27) and (28) in (29) and writing the parameter $\hat{m}_t$ used in (17) as $\hat{m}_t = \hat{m}(N_{t-1}, K_{t-1}, R_{t-1})$ the evolution of bond returns and cum-dividend prices may be expressed as

$$\begin{aligned} R_t &= \hat{\mathcal{R}}_\tau(\xi_{t-1}, \eta_t) := \mathcal{R}_\tau\Big(\hat{\mathcal{N}}(N_{t-1}), \hat{\mathcal{K}}(N_{t-1}, K_{t-1}, R_{t-1}), \hat{\mathcal{D}}(N_{t-1}, K_{t-1}, R_{t-1}) + \eta_t/\bar{x}, \\ &\qquad \mathcal{M}(\mu_{t-2}, q_{t-1}), z_{t-1}, R_{t-1}\Big) \\ q_t &= \hat{\mathcal{Q}}_\tau(\xi_{t-1}, \eta_t) := \frac{\pi(\mathcal{M}(\mu_{t-2}, q_{t-1}), \bar{\sigma}, \hat{m}(N_{t-1}, K_{t-1}, R_{t-1}))}{\hat{\mathcal{R}}_\tau(\xi_{t-1}, \eta_t)} \\ &\qquad + \hat{\mathcal{D}}(N_{t-1}, K_{t-1}, R_{t-1}) + \frac{\eta_t}{\bar{x}}. \end{aligned} \tag{31}$$

(c) *Dynamics of portfolios.* Combining (31) with (4) the wealth of each generation may be expressed as $w_t^{(j)} = w_\tau^{(j)}(\xi_{t-1}, \eta_t)$. Substituting this result together with (31) in the asset demand functions (6) the evolution of consumers' portfolios may be written as

$$\begin{aligned} x_t^{(j)} &= \hat{\varphi}_{x,\tau}^{(j)}(\xi_{t-1}, \eta_t), \quad j = 1, \ldots, J \\ y_t^{(j)} &= \hat{\varphi}_{y,\tau}^{(j)}(\xi_{t-1}, \eta_t), \quad j = 1, \ldots, J. \end{aligned} \tag{32}$$

□

Combining (26), (27), (28), (31) and (32) with the equivalent canonical representation $\{\eta \circ \vartheta^t\}_t$ of the process $\{\eta_t\}_t$ (cf. Appendix A.1) the evolution of the economy may for each fixed $\tau \in [0, 1[$ and initial state ξ_0 be written as

$$\xi_t = \phi_\tau(\xi_{t-1}, \eta(\vartheta^t(\tilde{\omega}))), \quad \tilde{\omega} \in \Omega,\ t \geq 0. \tag{33}$$

Equation (33) defines the time one-map of a random dynamical system in the sense of Arnold (1998) which provides the basis for the subsequent study. In particular, based on the random difference equation (33) we seek to study the long-run influence of changes in τ on consumer welfare.

In the sequel it will frequently be convenient to assume that the population is constant, i.e., $N_t^{(j)} \equiv \bar{N} > 0$. The justification for this is provided by the following lemma.

Lemma 1

Suppose that the parameters of the population law (20) satisfy $n^{(j)} \geq 0$, $j = 0, 1, \ldots, J$, $n_2 > 0$ and $\frac{1}{2} < n < 1$ where $n := \sum_{j=0}^{J} n^{(j)}$. Then the following holds true:

(i) The population dynamics (26) possess a unique positive steady state $N^\star = (\bar{N})_{j=0}^{J}$ where $\bar{N} = \frac{1}{(J+1)\,n_2} \ln \frac{n}{1-n}$.

(ii) The parameters $n^{(j)}$, $j = 0, \ldots, J$ can be chosen such that $N^\star$ is asymptotically stable for each value of $n_2 > 0$.

The assertion from Lemma 1 implies that *any* value $\bar{N} > 0$ can be induced as a stable steady state of (26) by suitably choosing the parameter n_2. This property will allow us to vary the population size $\bar{N}$ parametrically when studying the long-run properties of pension systems in the sequel.

3 Golden rules and long-run welfare

In this section we consider a simple parametrization of the model where $J = 1$, $j_L = 0$ such that there is one working and one retired generation. Let the population be constant with each generation consisting of $\bar{N} > 0$ consumers. The amount of labor supplied by young consumers is normalized to unity such that $\bar{L}^{(1)} \equiv 1$ and, therefore, $L_t^S \equiv \bar{N}$ in (1). In addition, setting $\varrho = 0$ in (24) we assume that first moment beliefs are constant such that $\mu_t \equiv \bar{\mu}$. Defining

$$\lambda_0 := \frac{\alpha\,\kappa \bar{N}^{\alpha}}{\bar{x}}\left[\frac{1-\alpha}{\alpha\,\gamma_1} + \tau\right] > 0 \quad \text{and} \quad \lambda_1 := \frac{1+\beta}{\beta}\frac{1}{1-\tau}\left[\frac{1-\alpha}{\alpha\,\gamma_1} + \tau\right] \tag{34}$$

it is shown in Hillebrand (2007) that the state dynamics of the model reduce to the evolution of capital and the bond return which take the form

$$\begin{aligned} K_t &= \hat{\mathcal{K}}(K_{t-1}, R_{t-1}; \bar{N}) &&:= \left[\frac{1}{\gamma_1}\ln\left(\frac{(1-\alpha)\kappa}{\gamma_0\gamma_1 R_{t-1}}\left(\frac{\bar{L^S}}{K_{t-1}}\right)^{\alpha}\right) + 1 - \delta\right] K_{t-1} \\ R_t &= \hat{\mathcal{R}}_\tau(K_{t-1}, R_{t-1}; \bar{N}, \bar{\mu}) &&:= \frac{\lambda_1\,\pi(\bar{\mu}, \bar{\sigma}, \lambda_0\,\hat{\mathcal{K}}(K_{t-1}, R_{t-1}; \bar{N})^{1-\alpha})}{\lambda_0\,\hat{\mathcal{K}}(K_{t-1}, R_{t-1}; \bar{N})^{1-\alpha}} + \lambda_1 - \frac{\tau}{1-\tau}. \end{aligned} \tag{35}$$

Equation (35) is a two-dimensional deterministic dynamical system whose properties with respect to τ and $\bar{N}$ are studied extensively in Hillebrand (2007). In particular, the existence of steady states along which consumers have rational expectations is investigated. Such steady states will be referred to as rational expectations equilibria (REE) to be introduced next. Here $\tilde{\eta}_t := \eta_t - \bar{\eta}$, $t \geq 0$ denotes the centered noise process from Assumption 1 with symmetric distribution $\nu_{\tilde{\eta}}$ supported on $[-\bar{\eta}, \bar{\eta}]$.

Definition 1
Let $J = 1$, $j_L = 0$ and the size $\bar{N} > 0$ of each generation and $\tau \in [0, 1[$ be fixed. A stationary rational expectations equilibrium is a triple $(K^\star, R^\star, \mu^\star)$ such that

(i) The pair $(K^\star, R^\star)$ is a steady state of (35) which is asymptotically stable.

(ii) The corresponding cum-dividend price process $\{q_t\}_{t\geq 0}$ defined by (31) converges point-wise to a stationary process $\{q_t^\star\}_{t\geq 0}$ where

$$q_t^\star = \mu^\star + \bar{\sigma}\tilde{\eta}_t. \tag{36}$$

The property in (ii) ensures that the (constant) mean forecast can be chosen to coincide with the mean of the cum-dividend price process along the steady state. Hence, setting $\bar{\sigma} = \bar{x}^{-1}$ and $\nu_\varepsilon = \nu_{\tilde{\eta}}$ it follows from (36) that the *perceived* distribution coincides with the objective distribution of cum-dividend prices along the steady state of (35). Moreover, the forecasting rules (21) and (22) imply correct forecasts for real wages and non-capital income along the steady state such that the hypothesized forecasting behavior indeed generates asymptotically rational expectations. Also note that by market

clearing of the stock and bond market young consumers' portfolios will be constant in the long run and given by $x_t^{(1)} \equiv \bar{x}/\bar{N}$ and $y_t^{(1)} \equiv \bar{B}/\bar{N}$. Here $\bar{B} := B(R^\star; \mathcal{W}(\bar{N}, K^\star), K^\star)$ is the firm's constant bond supply along the steady state.
The following result links the definition of a REE to the concept of a stable random fixed point which provides the basis for the subsequent welfare concept.

Lemma 2
Suppose $J = 1$, $j_L = 0$ and let $\tau \in [0, 1[$ be fixed. Suppose $N^\star = (\bar{N}, \bar{N})$ is the unique positive steady state of the population law (26) defined in Lemma 1 which is asymptotically stable. Let $(K^\star, R^\star, \mu^\star)$ be a REE in the sense of Definition 1 and define $\bar{B}$ as above. Then the random variable

$$\xi^\star(\tilde{\omega}) := (N^\star, K^\star, \mu^\star, \mu^\star, R^\star, \mu^\star + \bar{\sigma}\tilde{\eta}(\tilde{\omega}), \bar{x}/\bar{N}, \bar{B}/\bar{N}),\ \tilde{\omega} \in \Omega \tag{37}$$

is a stable random fixed point of the random dynamical system (33).

Observe that the only random component of the random variable $\xi^\star$ is the cum-dividend price which follows an i.i.d. process due to Assumption 1. Finding a stable random fixed point of the random dynamical system (33) therefore reduces to finding a REE. The following lemma states conditions under which a REE exists. The proof may be found in Hillebrand (2007).

Lemma 3
Let $J = 1$, $j_L = 0$ and the size $\bar{N} > 0$ of each generation be constant. Set $\bar{\sigma} = \bar{x}^{-1}$ and $\nu_\varepsilon = \nu_{\tilde{\eta}}$ for consumers' expectations and suppose that the production parameter in (8) satisfies $\alpha > \frac{1}{2}$. Then there exist differentiable mappings $(K^\star, R^\star, \mu^\star) : [0, 1[\longrightarrow \mathbb{R}_{++} \times]1, \infty[\times]\bar{\eta}/\bar{x}, \infty[, \tau \longmapsto (K^\star_\tau, R^\star_\tau, \mu^\star_\tau)$ which define the unique REE of (35) for each $\tau \in [0, 1[$. Moreover, $K^\star$ and $\mu^\star$ are strictly decreasing while $R^\star$ is strictly increasing.

For the following derivations we assume that the hypotheses of Lemma 3 are satisfied. In this case the triple $(K^\star_\tau, R^\star_\tau, \mu^\star_\tau)$ defines a REE of the dynamic equation (35) for each $\tau \in [0, 1[$. Hence, defining for each τ the random variable $\xi^\star_\tau$ as in (37) one obtains a unique stable random fixed point of (33). Given this result we are interested in the properties of the consumption processes along the random fixed point depending on τ. Utilizing the consumption function in (6), the definitions of non-capital income (2) and (3) and wealth (4), together with the wage law (16) and the forecasting rule (22) youthful consumption is deterministic and given by

$$c_\tau^{\star(1)}(\tilde{\omega}) = \frac{\alpha}{\bar{N}} \frac{\kappa \bar{N}^\alpha K_\tau^{\star\,1-\alpha}}{1+\beta} \left[1 - \tau + \frac{\tau}{R^\star_\tau}\right],\ \tilde{\omega} \in \Omega. \tag{38}$$

Likewise, using (4) and (6) and the canonical representation of the noise process old age consumption along the random fixed point follows a stationary process $\{c_\tau^{\star(0)} \circ \vartheta^t\}_{t \geq 0}$ corresponding to the random variable

$$c_\tau^{\star(0)}(\tilde{\omega}) = \frac{1}{\bar{N}} \left[\kappa \bar{N}^\alpha K_\tau^{\star\,1-\alpha}\Big(\alpha\tau + \frac{1-\alpha}{\gamma_1}\Big) + \bar{x}\,\mu^\star_\tau + \tilde{\eta}(\tilde{\omega})\right],\ \tilde{\omega} \in \Omega. \tag{39}$$

It follows from (5) that for each τ the lifetime utility process along the random fixed point takes the form of an ergodic stationary process $\{U_\tau^\star \circ \vartheta^t\}_{t\geq 1}$ where

$$U_\tau^\star(\tilde\omega) := \ln c_\tau^{\star(1)}(\tilde\omega) + \beta \ln c_\tau^{\star(0)}(\tilde\omega),\ \tilde\omega \in \Omega.$$

The expected lifetime utility of consumers along the steady state will therefore be constant and identical for all generations taking the form

$$\begin{aligned}\mathbb{E}[U_\tau^\star] &= \ln\Big[\frac{\alpha}{\bar N}\frac{\kappa\,\bar N^\alpha K_\tau^{\star\,1-\alpha}}{1+\beta}\big(1-\tau+\frac{\tau}{R_\tau^\star}\big)\Big] \\ &\quad +\beta\int_{[-\bar\eta,\bar\eta]} \ln\Big[\frac{1}{\bar N}\Big(\kappa\,\bar N^\alpha K_\tau^{\star\,1-\alpha}\big(\alpha\tau+\frac{1-\alpha}{\gamma_1}\big)+\bar x\,\mu_\tau^\star+\tilde\eta\Big)\Big]\nu_{\tilde\eta}(d\tilde\eta).\end{aligned} \tag{40}$$

The following result characterizes the expected lifetime utility depending on the contribution rate. The proof may be found in Appendix A.3.

Theorem 1
Let the hypotheses of Lemma 3 be satisfied. Then the map $\tau \longmapsto \mathbb{E}[U_\tau^\star]$ defined in (40) is strictly decreasing. Hence, any reduction in the contribution rate leads to a long-run welfare improvement of consumers.

Theorem 1 strongly suggests an inefficiency of a pure pay-as-you-go system in the presence of an active asset and bond market. It implies that in the long run any reduction in τ is favorable and leads to higher welfare of generations. It will be demonstrated in the following section that this result continues to hold even for a more general parametrization of the model.

While the previous result required that $\alpha > \frac{1}{2}$, a more general argument proceeds as follows. Since the following derivations are valid also in the multi-period case we let for the moment $J \geq 1$ be arbitrary. Let $C_t := \sum_{j=0}^J N_t^{(j)} c_t^{(j)}$ denote aggregate consumption and $S_t := \sum_{j=0}^J N_t^{(j)} y_t^{(j)} = B_t$ aggregate savings corresponding to consumers' net bond demand at time t. Then the definition of wealth (4) together with (6) and (12) and asset market clearing imply for each t the aggregate identity

$$F(L_t^S, K_t, \eta_t) = C_t + S_t. \tag{41}$$

Now suppose that the population is constant implying that $L_t^S \equiv \bar L^S$ and assume that the capital dynamics are at some steady state $\bar K > 0$. In this case, investment I_t will necessarily be constant and equal to $I_t \equiv \bar I = \delta \bar K$. Hence, using the adjustment cost function (10) savings will be constant as well and satisfy $S_t \equiv \bar S := \bar K g(\delta)$. Using this in (41) shows that aggregate consumption at time t is given by

$$\begin{aligned} C_t = C(\bar K, \bar L^S, \eta_t) &:= F(\bar L, \bar K, \eta_t) - g(\delta)\bar K \\ &= \kappa \bar L^{S\,\alpha}\bar K^{1-\alpha} + \eta_t - \gamma_0 \exp\{\gamma_1\,\delta\}\bar K.\end{aligned} \tag{42}$$

Observe that the map $K \longmapsto C(K, \bar L^S, \eta_t)$ is strictly concave and that the maximizer

$$K^{opt} := \arg\max\Big\{C(K, \bar L^S, \eta_t)|K \geq 0\Big\} = \Big[\frac{(1-\alpha)\kappa \bar L^{S\,\alpha}}{\gamma_0}\exp\{-\gamma_1\delta\}\Big]^{\frac{1}{\alpha}} \tag{43}$$

does not depend on η_t. The value K^{opt} will be referred to as the *golden rule capital stock*. It defines the capital stock at which aggregate consumption is maximal. For the case with no adjustment costs where $g(\delta) = \delta$ one recovers the classical result $\frac{\partial}{\partial K}F(\bar{L}^S, K^{opt}, \eta_t) = n + \delta = \delta$ noting that the population growth rate is $n = 0$.
Now suppose that the capital dynamics defined by (27) are at some steady state $\bar{K} > 0$. Note from (27) that this requires that bond returns are constant and equal to $\bar{R} > 0$. By a slight abuse of terminology the pair $(\bar{K}, \bar{R})$ will be referred to as a steady state of the capital dynamics (27). The following lemma relates this steady state to the golden rule value defined in (43).

Lemma 4
Let the pair $(\bar{K}, \bar{R})$ be a steady state of the capital dynamics (27). Then one has

$$\bar{K} \gtreqless K^{opt} \quad \text{if and only if} \quad \bar{R} \lesseqgtr \frac{1}{\gamma_1}.$$

Since $\frac{1}{\gamma_1} < \delta$, this implies that the critical value in Lemma 4 satisfies $\frac{1}{\gamma_1} < 1$. Hence, whenever $\bar{R} > 1$, the steady state will suffer from under-accumulation of capital.
Consider now the deterministic two-period case where $J = 1$ and $j_L = 0$ studied at the beginning of this section. While it can be shown that if $\alpha \leq \frac{1}{2}$ a REE may well fail to exist, it is shown in Hillebrand (2007) that once it does exist it will have $R^\star > 1$. Hence, we obtain as a consequence of Lemma 4 the following result.

Theorem 2
Let $J = 1$, $j_L = 0$ and let $\bar{N} > 0$ and $\tau \in [0, 1[$ be fixed. Suppose there exists a REE $(K^\star, R^\star, \mu^\star)$ in the sense of Definition 1. Then $K^\star < K^{opt}$.

If $\alpha > \frac{1}{2}$ the capital stock at the unique REE is determined by the map $\tau \longmapsto K^\star_\tau$ defined in Lemma 3. Since this map is strictly decreasing this result shows that any reduction of τ brings the steady state capital stock $K^\star_\tau$ closer to the golden rule value K^{opt}. This explains why a reduction in τ is welfare improving as asserted by Theorem 1. Nevertheless, even for $\tau = 0$ this value fails to coincide with the optimal size according to the golden rule. In the opposite case where $\alpha \leq \frac{1}{2}$ one can show that Theorem 1 continues to hold whenever a REE exists and a marginal reduction in τ leads to an increase of the capital stock.

4 Stationary welfare under the pension system

We are now in a position to extend the welfare concept motivated in the previous section to the general case where $J \geq 1$ and possibly $\varrho \neq 0$. For this purpose, let $\xi_0 \in \Xi$ denote the initial state of the system taken from some suitable set $\Xi \subset \mathbb{R}^{J+1}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}^J \times \mathbb{R}^J_+$. As before, let the contribution rate be constant such that $\tau_t \equiv \tau \in [0, 1[$. Using the dynamic equation (33) Appendix A.1 shows that for each

$\tilde{\omega} \in \Omega$ defining the path of the noise and initial state $\xi_0 \in \Xi$ the dynamic evolution can be written as a measurable flow $\Phi_\tau : \mathbb{N}_0 \times \Omega \times \Xi \longrightarrow \Xi$

$$\xi_t = \Phi_\tau(t, \tilde{\omega}, \xi_0) := \begin{cases} \xi_0 & t = 0 \\ \phi_\tau\left(\eta(\vartheta^t \tilde{\omega})\right) \circ \ldots \circ \phi_\tau\left(\eta(\vartheta \tilde{\omega})\right) \xi & t \geq 0. \end{cases} \tag{44}$$

To simplify our notation we write the components of the process $\{\Phi_\tau(t, \cdot, \xi_0)\}_{t\geq 0}$ as $\{K_t(\tau, \cdot, \xi_0)\}_{t\geq 0}$, $\{q_t(\tau, \cdot, \xi_0)\}_{t\geq 0}$, etc. The process $\{\Phi_\tau(t, \cdot, \xi_0)\}_{t\geq 0}$ defines an induced consumption process $\{c_t(\tau, \cdot, \xi_0)\}_{t>0}$ where $c_t(\tau, \cdot, \xi_0) = \left(c_t^{(j)}(\tau, \cdot, \xi_0)\right)_{j=0}^J : \Omega \longrightarrow \mathbb{C}^{J+1}$ describes individual consumption in generations $j \in \{0, \ldots, J\}$ at time t. Combining the consumption function in (6) with the definition of wealth (4) and the forecasting rules (22) and (23) yields the components of the random variable $c_t(\tau, \cdot, \xi_0)$ as[2]

$$\begin{aligned} c_t^{(j)}(\tau, \tilde{\omega}, \xi_0) \;&:=\; \bar{c}^{(j)}\Big(e_t^{(j)}(\tau, \tilde{\omega}, \xi_0) + R_{t-1}(\tau, \tilde{\omega}, \xi_0)\, y_{t-1}^{(j+1)}(\tau, \tilde{\omega}, \xi_0) \\ &+\; q_t(\tau, \tilde{\omega}, \xi_0)\, x_{t-1}^{(j+1)}(\tau, \tilde{\omega}, \xi_0) + \frac{1}{R_t(\tau, \tilde{\omega}, \xi_0)} \sum_{m=1}^{j} \frac{e_t^{(j-m)}(\tau, \tilde{\omega}, \xi_0)}{R_{t-1}(\tau, \tilde{\omega}, \xi_0)^{m-1}}\Big), \; \tilde{\omega} \in \Omega \end{aligned} \tag{45}$$

for each $t > 0$ and $j = 0, \ldots, J$. Here for each $\tilde{\omega} \in \Omega$

$$e_t^{(j)}(\tau, \tilde{\omega}, \xi_0) := \begin{cases} \dfrac{\tau\, \mathcal{W}(K_t(\tau, \tilde{\omega}, \xi_0), L_t^S)\, L_t^S}{\sum_{j=0}^{j_L - 1} N_t^{(j)}} & j = 0, \ldots, j_L - 1 \\ (1 - \tau)\, \mathcal{W}(K_t(\tau, \tilde{\omega}, \xi_0), L_t^S)\, \bar{L}^{(j)} & j = j_L, \ldots, J \end{cases} \tag{46}$$

denotes the induced non-capital income process with L_t^S being defined as in (1) and the map $\mathcal{W}$ as in (16). Associated with the consumption process $\{c_t(\tau, \cdot, \xi_0)\}_{t>0}$ is the induced lifetime utility process $\{U_t(\tau, \cdot, \xi_0)\}_{t>J}$ where for each $t > J$ the random variable $U_t(\tau, \cdot, \xi_0) : \Omega \longrightarrow \mathbb{R}$ is defined as

$$U_t(\tau, \tilde{\omega}, \xi_0) := \sum_{j=0}^{J} \beta^{J-j} \ln c_{t-j}^{(j)}(\tau, \tilde{\omega}, \xi_0), \quad \tilde{\omega} \in \Omega. \tag{47}$$

For each $t > J$ the quantity in (47) describes the lifetime utility attained by the consumers who were born in period $t - J$.

Now assume that for each $\tau \in [0, \bar{\tau}]$ the system (44) possesses a stable random fixed point (cf. Appendix A.1). In this case, the long-run behavior of the system is described by the ergodic process $\{\xi_\tau^\star \circ \vartheta^t\}_{t\geq 0}$. The idea of the following efficiency concept is to compare the lifetime utilities attained along the path of the random fixed point induced by different contribution rates τ. To formalize this idea, recall that along the random fixed point one has $N_t^{(j)} \equiv \bar{N}$ and, using (1), $L_t^s \equiv \bar{L}^S$. Write the map $\xi_\tau^\star : \Omega \longrightarrow \Xi$ in component form $\xi_\tau^\star(\cdot) = (N^\star(\cdot), K_\tau^\star(\cdot), \mu_\tau^\star \circ \vartheta^{-1}(\cdot), \mu_\tau^\star(\cdot), R_\tau^\star(\cdot), q_\tau^\star(\cdot), y_\tau^\star(\cdot), x_\tau^\star(\cdot))$ where $N^\star(\cdot) \equiv (\bar{N})_{j=0}^J$, $y_\tau^\star(\cdot) = (y_\tau^{(j)\star}(\cdot))_{j=1}^J$ and $x_\tau^\star(\cdot) = (x_\tau^{(j)\star}(\cdot))_{j=1}^J$. Then each choice of

[2] We define $y_{t-1}^{(j+1)}(\tau, \cdot, \xi_0) \equiv 0$ and $x_{t-1}^{(j+1)}(\tau, \cdot, \xi_0) \equiv 0$ for $j = J$ and $\sum_{m=1}^{j} \frac{e_t^{(j-m)}(\tau, \cdot, \xi_0)}{R_{t-1}(\tau, \cdot, \xi_0)^{m-1}} \equiv 0$ for $j = 0$. Similar conventions apply in (48).

$\tau \in [0, \bar{\tau}]$ gives rise to an induced process $\{c^\star_\tau \circ \vartheta^t\}_{t>0}$ describing consumption of generations along the random fixed point $\{\xi^\star_\tau \circ \vartheta^t\}_{t\geq 0}$. Utilizing the same procedure as in the derivation of (45) the components of $c^\star_\tau = (c^{(j)\star}_\tau)^J_{j=0} : \Omega \longrightarrow \mathbb{C}^{J+1}$ take the form

$$\begin{aligned} c^{(j)\star}_\tau(\tilde{\omega}) &= \bar{c}^{(j)} \Big(e^{(j)\star}_\tau(\tilde{\omega}) + R^\star_\tau(\vartheta^{-1}\tilde{\omega}) y^{(j+1)\star}_\tau(\vartheta^{-1}\tilde{\omega}) + q^\star_\tau(\tilde{\omega})\, x^{(j+1)\star}_\tau(\vartheta^{-1}\tilde{\omega}) \\ &\quad + \frac{1}{R^\star_\tau(\tilde{\omega})} \sum_{m=1}^{j} \frac{e^{(j-m)\star}_\tau(\tilde{\omega})}{R^\star_\tau(\vartheta^{-1}\tilde{\omega})^{m-1}} \Big) \end{aligned} \tag{48}$$

where for each $\tilde{\omega} \in \Omega$, letting $\bar{L}^S := \sum_{j=j_L}^J \bar{L}^{(j)} \bar{N}$

$$e^{(j)\star}_\tau(\tilde{\omega}) := \begin{cases} \dfrac{\tau\, \mathcal{W}(K^\star_\tau(\tilde{\omega}), \bar{L}^S) \bar{L}^S}{\sum_{j=0}^{j_L-1} \bar{N}} & j = 0, \dots, j_L - 1 \\ (1-\tau)\, \mathcal{W}(K^\star_\tau(\tilde{\omega}), \bar{L}^S) \bar{L}^{(j)} & j = j_L, \dots, J. \end{cases}$$

The consumption process $\{c^\star_\tau \circ \vartheta^t\}_{t>0}$ is again ergodic and gives rise to an induced utility process $\{U^\star_\tau \circ \vartheta^t\}_{t>J}$ describing lifetime utility attained by consumers along the path of the random fixed point. The random variable $U^\star_\tau : \Omega \longrightarrow \mathbb{R}$ is defined as

$$U^\star_\tau(\tilde{\omega}) := \sum_{j=0}^{J} \beta^{J-j} \ln c^{(j)\star}_\tau(\vartheta^{-j}\tilde{\omega}). \tag{49}$$

The process $\{U^\star_\tau \circ \vartheta^t\}_{t>J}$ inherits the properties of ergodicity from the consumption process $\{c^\star_\tau \circ \vartheta^t\}_{t>0}$ and is hence stationary. This implies that the expected value $\mathbb{E}[U^\star_\tau \circ \vartheta^t] := \int_\Omega U^\star_\tau \circ \vartheta^t(\tilde{\omega}) \mathbb{P}(d\tilde{\omega})$ is independent of t such that $\mathbb{E}[U^\star_\tau] = \mathbb{E}[U^\star_\tau \circ \vartheta^t]$ for all t. For each $\tau \in [0, \bar{\tau}]$ the real number $\mathbb{E}[U^\star_\tau]$ describes the expected lifetime utility attained by consumers along the random fixed point. The efficiency criterion which will be used in the sequel will be to choose $\tau \in [0, \bar{\tau}]$ such that $\mathbb{E}[U^\star_\tau]$ becomes maximal. For this purpose, the following proposition shows that stability of the random fixed point implies that the paths of the consumption and utility processes converge to the corresponding paths of consumption and utility along the random fixed point.

Lemma 5
Let $\tau \in [0, \bar{\tau}]$ be arbitrary and suppose that the random dynamical system (44) possesses a stable random fixed point. Then both the consumption process $\{c^\star_\tau \circ \vartheta^t\}_{t>0}$ defined by (48) and the lifetime utility process $\{U^\star_\tau \circ \vartheta^t\}_{t>J}$ defined by (49) are stable in the sense that for each $j = 0, \dots, J$

$$(i)\ \lim_{t\to\infty} \|c^{(j)\star}_\tau(\vartheta^t\tilde{\omega}) - c^{(j)}_t(\tau, \tilde{\omega}, \xi_0)\| = 0 \quad \textit{and} \quad (ii)\ \lim_{t\to\infty} \|U^\star_\tau(\vartheta^t\tilde{\omega}) - U_t(\tau, \tilde{\omega}, \xi_0)\| = 0$$

for all $\xi_0 \in U(\tilde{\omega})$ $\mathbb{P}$-a.s.

Exploiting ergodicity of the process $\{U^\star_\tau \circ \vartheta^t\}_{t>J}$ and the stability result from Lemma 5, the expected utility along the random fixed point may again be obtained from time averages such that

$$\mathbb{E}\left[U_\tau^\star\right] = \lim_{T\to\infty} \frac{1}{T-J} \sum_{t=J+1}^{T} U_t(\tau, \tilde{\omega}, \xi_0) \quad \mathbb{P}-a.s.$$

In the sequel the expected value $\mathbb{E}\left[U_\tau^\star\right]$ will again be approximated by the sample mean

$$\hat{\mathbb{E}}\left[U_\tau^\star\right] = \frac{1}{T-J} \sum_{t=J+1}^{T} U_t(\tau, \tilde{\omega}, \xi_0).$$

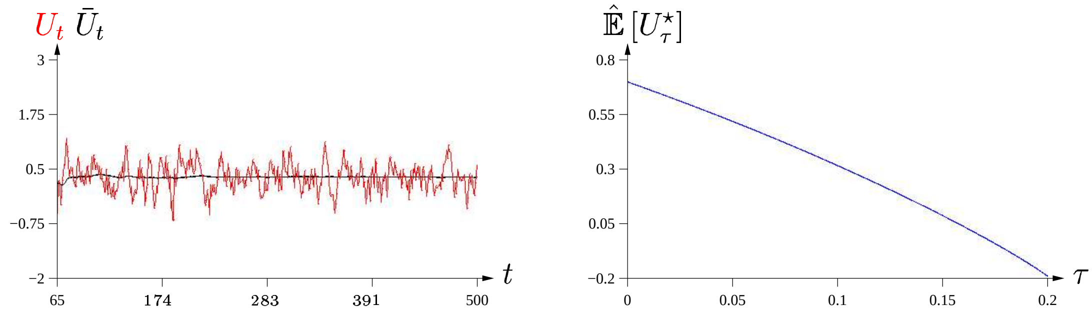
The remainder of this paper presents results from numerical simulations using a calibrated parametrization of the model which is justified on the grounds of empirical studies. We first study the long-run welfare properties of pension systems by employing the concept developed in this section. In this part we assume that the population is constant, i.e., $N_t^{(j)} \equiv \bar{N}$. The second part considers the case with demographic transition periods. The employed parameter values are listed in Table 1.

Parameter	Value	Description	Parameter	Value	Description
J	14	Life expectancy	κ	2.5	Production parameter
j_L	6	Retired generations	γ_0	0.02	Adjustment cost parameter
$\bar{N}$	1000	Consumers per gen.	γ_1	7.5	Adjustment cost parameter
$\bar{L}^{(j)}$	1	Individual labor supply	δ	0.28	Rate of depreciation
β	0.96	Discount factor	$\bar{x}$	5000	Total number of shares
$\bar{\varepsilon}$	0.92	Expectations parameter	η_{max}	2000	Upper bound for real noise
ϱ	0.5	Expectations parameter	K_0	4,500	Initial capital stock
σ	0.96	Dispersion parameter	q_0	9.5	Initial cum-dividend price
α	0.65	Production parameter	R_0	1.12	Initial bond return

Table 1: Standard parameter set for the numerical simulations

The application of the welfare concept developed in this section requires the existence of a stable random fixed point. While Lemma 3 provides an existence result for the two-period case with constant beliefs for asset prices, a generalization to the stochastic multi-period case seems difficult. In Appendix A.7 we therefore establish the existence of a stable random fixed point numerically for the given parametrization and $\tau \in [0, 0.2]$ such that our efficiency concept becomes applicable. In particular, the long-run behavior of the model is independent of the initial state $\xi_0 \in \Xi$ and convergence to the path of the random fixed points obtains within the first fifty periods as the initial state is varied.

To alleviate the following notation we shall frequently suppress the dependence of random variables on arguments writing e.g. U_t as a shorthand for the realization $U_t(\tau, \tilde{\omega}, \xi_0)$. Figure 1 depicts the lifetime utility process and its expected value depending on contribution rates $\tau \in [0, 0.2]$. The left figure shows the utility process defined by (47) together with the recursive mean for the intermediate case $\tau = 0.1$. The right hand side presents a bifurcation plot showing the corresponding sample mean $\hat{\mathbb{E}}\left[U_\tau^\star\right]$ for alternative values $\tau \in [0, 0.2]$. Figure 1 suggests a strictly negative relationship between the

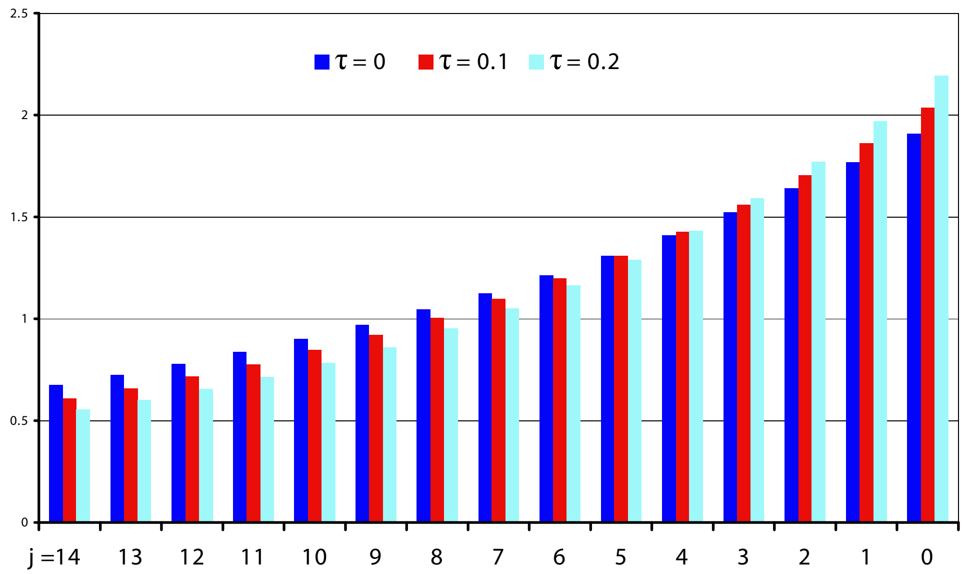


(a) Lifetime utility process, $\tau = 0.1$ (b) $\hat{\mathbb{E}}(U^\star_\tau)$ depending on τ

Figure 1: Impact of contribution rates on consumer welfare

contribution rate τ and the sample mean $\hat{\mathbb{E}}[U^\star_\tau]$. By ergodicity of the utility process this suggests that long-run expected utility $\mathbb{E}[U^\star_\tau]$ will be higher as the contribution rate τ is reduced. Hence, *any* reduction of the public pension system will lead to a long-run welfare improvement of consumers. Conversely, in the present scenario of a stationary population, any persistent increase in τ will reduce the long-run welfare of all generations. This result confirms the assertion from Theorem 1 for the general case.

Next consider how a change in τ affects the distribution of consumption over the life cycle. Figure 2 compares the average consumption of consumers during their life-cycle for three cases where $\tau \in \{0, 0.1, 0.2\}$. For each period $j \in \{0, 1, \ldots, J\}$ of the life cycle the values are calculated as averages of the sample $\{c_t^{(j)}\}_{t=50}^{500}$. By ergodicity, these values approximate the expected value of the consumption process $\{c_t^{(j)}(\tau, \cdot, \xi_0)\}_{t>0}$.



Average consumption in	Working age			Retirement age		
Contrib. rate	$\tau = 0$	$\tau = 0.1$	$\tau = 0.2$	$\tau = 0$	$\tau = 0.1$	$\tau = 0.2$
Mean consumption	8.27	7.83	7.33	9.56	9.90	10.25

Figure 2: Average consumption over the life-cycle; $\tau \in \{0, 0.1, 0.2\}$

In all three cases expected consumption strictly increases with age and is thus least when young and largest when old. While this result is qualitatively independent of the contribution rate, a reduction in τ changes the distribution by shifting more consumption towards the earlier periods of the life cycle. This implies that a reduction in τ fosters consumption in working age and reduces consumption when retired. Comparing expected consumption during the working years and the retirement age the table below shows that even for a contribution rate $\tau = 0$ retired consumers have higher consumption than consumers in working age. An increase in contributions broadens this gap by increasing consumption when retired while reducing it during the working years. In this regard, recall that the retirement period corresponds to six periods of the life-cycle while the working period is nine periods long. A reduction in τ thus leads to a more uniform distribution of consumption over the life cycle which, as shown before, has a positive impact on expected lifetime utility and increases the welfare of consumers.

It is demonstrated in Hillebrand (2006) that all of the previous results are robust against parameter changes. They unequivocally suggest that the installation of a pension system is not desirable and should be avoided. Conversely, if the pension system is already installed, any reduction of contributions will lead to a welfare improvement of generations in the long run. Hence, from a long-run perspective, any social authority that controls the contribution rate should seek to abandon the pension system by ultimately reducing contributions to zero. Note, however, that any reduction in contributions will lead to a loss of welfare on the part of current retirees. Hence, from a political point of view, the important issue is how a transition towards a lower contribution rate should be organized such that these losses remain sufficiently small. This issue will be discussed in the following section.

5 Reducing contributions

Consider the case where initially a public pension system exists which is characterized by a constant contribution rate $\tau_t \equiv 0.2$. Assume that the long-run goal of the social authority which controls the pension system is to reach a contribution rate $\tau_t \equiv 0$. To achieve this goal, the present section discusses two different scenarios. In the first one, the pension system is immediately abandoned while it is gradually reduced in the second one.[3] More specifically, assume that the adjustment starts in period $t = 51$ such that $\tau_t = 0.2$ for $t \leq 50$. In the first scenario, where the system is instantaneously abandoned we have $\tau_t = 0$ for $t \geq 51$. This type of adjustment is denoted symbolically as $\tau_t = 0.2 \downarrow 0$. In the second scenario, we assume that from $t = 51$ onwards the contribution rate is gradually lowered by 0.5% points in every period. This implies that the target value of $\tau_t = 0$ is reached in period $t = 90$. The gradual adjustment policy is denoted as $\tau_t = 0.2 \searrow 0$. Figure 3 visualizes the adjustment policies for either scenario.

[3] Clearly, from a political point of view an instantaneous abandonment of the pension system seems unrealistic since it implies that some generations do not receive any pension payments although they have paid contributions. Nevertheless this scenario provides an interesting benchmark case.

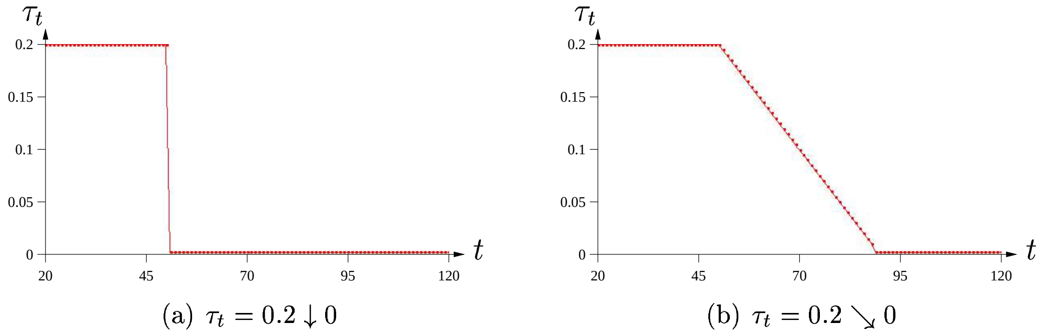


Figure 3: Instantaneous and gradual reduction of contribution rates

To display the consequences of either adjustment policy on consumer welfare we compare the expected utilities attained by the different generations during the transition phase. Since the utility process is non-stationary during the transition phase we draw a sample $(\tilde{\omega}^{(k)})_{k=1}^{K}$ of K different realizations of the noise process. Exploiting the law of large numbers one has for each t

$$\mathbb{E}\left[U_t(\tau, \cdot, \xi_0)\right] = \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} U_t(\tau, \tilde{\omega}^{(k)}, \xi_0).$$

In the sequel we approximate the limit by taking averages over $K = 450$ realizations of the noise process. The result is shown in Figure 4 which depicts the expected lifetime utilities attained by consumers during the transition phase $t \in \{45, \ldots, 110\}$.

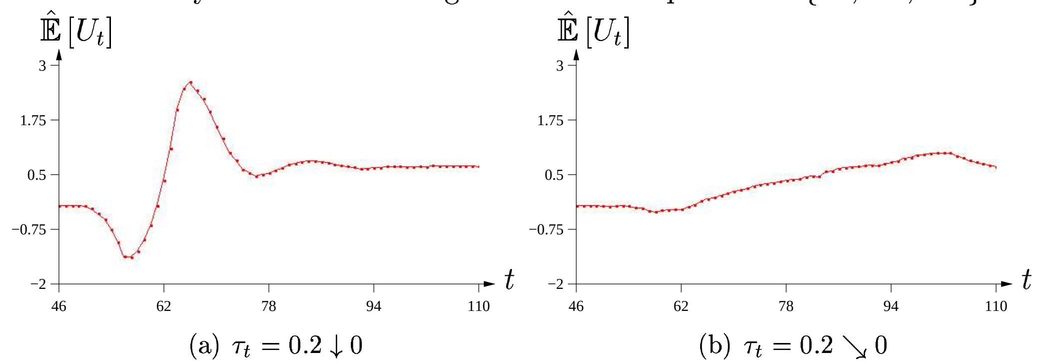


Figure 4: Impact of a reduction in contribution rates on expected lifetime utilities

For the instantaneous adjustment policy one observes a dramatic loss in utility during the time window $t \in \{51, \ldots, 56\}$. These are the generations which lose their pension income but do not benefit from the reduction in contribution rates during their working years. From $t = 57$ utility starts to increase again. These are now the generations which have increasingly benefitted from the reduced contribution rate during their working years. In this regard, recall that the reduction in contributions causes a shift of utility from the retirement age to the working age. After the initial decline we observe a large overshooting in utility which reaches a maximum in $t = 66$. After the peak utility decreases again to reach its long-run level from $t = 85$ onwards.
In contrast to this, the initial downturn as well as the following overshooting is almost entirely avoided with a gradual reduction in contribution rates. One also observes that

with the gradual adjustment policy all generations are better off from $t = 65$ while with an instantaneous adjustment this is the case from $t = 62$ onwards. Nevertheless, from a political point of view, the gradual reduction of contributions appears to be much more favorable since it generates a much smoother transition and improves the welfare of all generations in the long run.

6 Demographic transition

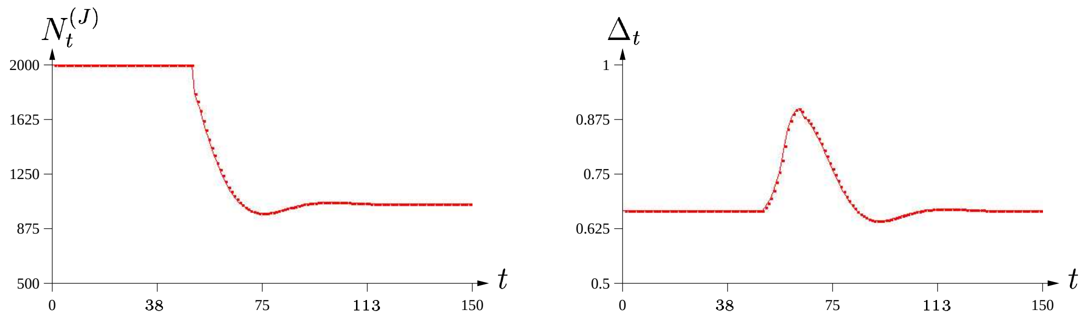
The following sections extend the previous simulation study to the case with demographic change and a shrinking population. Here, demographic change is modelled as a transitory phenomenon due to a (permanent) shift in the steady state of the population dynamics. Recall from Lemma 1 that the parameter n_2 is crucial to determine the steady state value $\bar{N}$ while it does not affect its dynamic stability. This property allows us to vary the value $\bar{N}$ by varying n_2 without affecting its asymptotic stability.
For the following numerical investigation we assume the map $j \longmapsto n_0^{(j)}$ to be non-decreasing. Recalling that a consumer's life starts at the age of 20 ($j = J$) and ends at the age of 80 ($j = 0$) with each period corresponding to four years this property reflects the plausible assumption that fertility is a decreasing function of age. The parameters used in the subsequent simulations are summarized in the following table.[4]

Parameter	Value	Description	Parameter	Value	Description
J	14	Number of generations	$n_1^{(11)}$	0.1	Fertility (age 32-35)
$n_0^{(14)}$	0.275	Fertility (age 20-23)	$n_0^{(10)}$	0.05	Fertility (age 36-39)
$n_0^{(13)}$	0.25	Fertility (age 24-27)	$n_0^{(9)}$	0.01	Fertility (age 40-43)
$n_0^{(12)}$	0.2	Fertility (age 28-31)	$n_0^{(j)}$, $j \leq 8$	0	Fertility (age ≥ 44)

Table 2: Parameter values for the population dynamics.

The parameter choices in Table 2 induce an asymptotically stable steady of the population dynamics (26) for each choice $n_2 > 0$. To model the demographic transition we assume that initially ($t \leq 50$) the population is in a steady state such that $N_t \equiv (\bar{N})_{j=0}^{J}$ where $\bar{N} \approx 2000$ corresponding to a parameter choice $n_2 = 0.000067$. In period $t = 51$ the parameter n_2 changes to $n_2 = 0.00013$ shifting the steady state to a lower value $\bar{N}' \approx 1000$. The associated adjustment process of the population towards the new steady state value then defines a demographic transition period during which the number of births decreases. Note that the steady state value $\bar{N}$ corresponds to the value used in the simulations of the previous sections. Hence the previous results remain valid in the long run as soon as the population has reached the new steady state. The evolution of the population represented by the number of births as well as the associated (economic) dependency ratio $\Delta_t := \sum_{j=0}^{j_L-1} N_t^{(j)} / \sum_{j=j_L}^{J} N_t^{(j)}$ is depicted in Figure 5.

[4] At this point neither the population model nor the parameter choices are justified on empirical grounds. For our purpose the prescribed specifications offer a simple way to model demographic transitions of the population and to study the implications for the pension system.

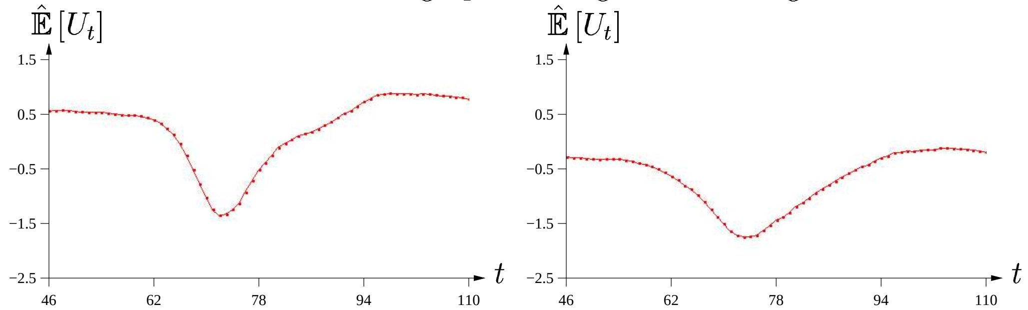


(a) Number of young consumers (b) Dependency ratio

Figure 5: Time profiles under demographic transition

Figure 5(a) shows that the level of births convergence to its new steady state value within slightly less than 50 periods. As a consequence it can be shown that for $t \geq 110$ the population will be constant again such that $N_t \equiv (\bar{N}')_{j=0}^{J}$ and the dynamic behavior is as described in the previous sections. In the sequel we will therefore mostly restrict attention to the demographic transition period where $t \in \{51, \ldots, 110\}$. As is seen from Figure 5(b) demographic transition is accompanied by a dramatic temporary increase in the dependency ratio which reaches a maximum of $\approx 90\%$ in $t = 63$, i.e. after 12 periods ($\approx$ 50 years) before it eventually returns to its initial value of 66%. This range corresponds roughly to the predicted evolution for the German population over the next 50 years as exhibited in Börsch-Supan, Heiss, Ludwig & Winter (2003). Thus our demographic model mimics roughly the predicted demographic change of the German population over the next 50 years.

Consider now the impact of demographic change on the welfare of consumers for the case with a constant contribution rate $\tau_t \equiv \tau$. For simplicity, we restrict attention to the boundary cases where $\tau \in \{0, 0.2\}$. The result is displayed in Figure 6 showing the expected lifetime utilities of generations during the demographic transition for both scenarios. One observes that demographic change causes a significant decline in con-



(a) $\tau = 0$ (b) $\tau = 0.2$

Figure 6: Profiles of expected lifetime utilities under demographic change

sumer welfare which mirrors - with a slight delay - the evolution in the dependency ratio (cf. Figure 5(b)). This decline occurs independently of the prevailing contribution rate and is only slightly attenuated if the contribution rate is lower (although expected utility remains higher with a lower contribution rate throughout the entire time window). After the demographic transition expected utility returns again to a stationary level.

These observations suggest that without further adjustments of the pension system, demographic change has a serious impact on consumers' welfare resulting in large welfare losses during the demographic transition period.

7 Adjusting contributions during transition

The previous result raises the natural question how the pension system should be adjusted to avoid or at least attenuate these effects. The most straightforward adjustment is a change in the contribution rate during the transition which will be studied in this section. In this regard, consider the case where pension payments and contributions at time t are determined by the adjustment formula

$$\begin{cases} e_t^R &= e_{t-1}^R \, \frac{\omega_{t-1}}{\omega_{t-2}} \, \frac{b^{(1)} - \tau_{t-1}}{b^{(1)} - \tau_{t-2}} \Big(1 + b^{(2)} \left(1 - \frac{\Delta_{t-1}}{\Delta_{t-2}}\right)\Big) \\ \tau_t &= \frac{e_t^R \, N_t^R}{\omega_t \, L_t^S} \end{cases} \tag{50}$$

where $0 < b^{(1)} \leq 1, \; 0 \leq b^{(2)} \leq 1$. The adjustment policy (50) determines the pension income e_t^R at time t essentially from three determinants while contributions τ_t are adjusted accordingly. The first factor $\frac{\omega_{t-1}}{\omega_{t-2}}$ accounts for the previous change in gross real wages. The second factor $\frac{b^{(1)}-\tau_{t-1}}{b^{(1)}-\tau_{t-2}}$ captures changes in the contribution rate over the last two periods where the parameter $b^{(1)}$ is assumed to be sufficiently close to unity such that $b^{(1)} > \tau_t$ for all times t. Note that a decrease in $b^{(1)}$ will increase the sensitivity to changes in previous contribution rates. Finally, the third factor $1 + b^{(2)} \left(1 - \frac{\Delta_{t-1}}{\Delta_{t-2}}\right)$ accounts for demographic changes of the population measured by the previously observed change in the dependency ratio. With this specification an increase in real wages has a positive impact on current pensions while a previous increase in contribution rates has a diminishing impact. Likewise an increase in the factor $\frac{\Delta_{t-1}}{\Delta_{t-2}}$ corresponding to an accelerated aging of the population will decrease the pension income at time t.
Given the general form of the adjustment formula (50) the present section studies four different pension policies corresponding to different parameter choices $b^{(1)}$ and $b^{(2)}$ in (50). A more detailed description of these policies and their application to the German pension system can be found in Rürup (2003), pp. 98 and Börsch-Supan, Heiss, Ludwig & Winter (2003) who conduct a study in the same spirit.

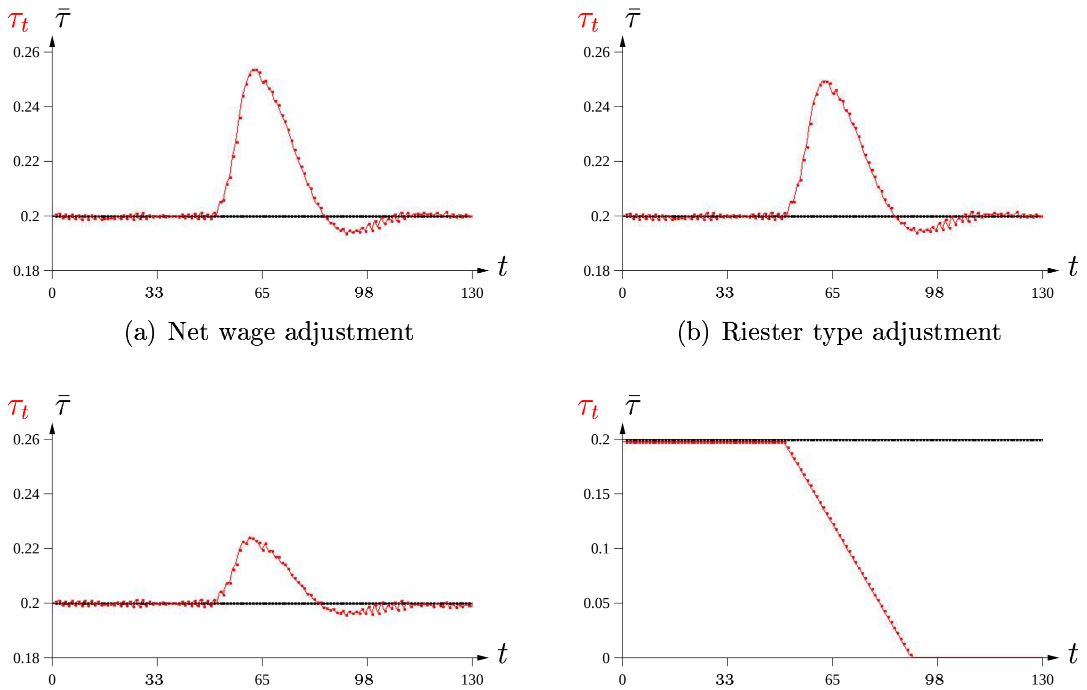
Case 1: Net Wage Adjustment ($b^{(1)} = 1$, $b^{(2)} = 0$)
With these parameter choices one observes from (50) that the growth rate of pension incomes is determined from the previous increase in net wages. Consequently, this adjustment policy is referred to as net wage adjustment. Note that the demographic change of the population does not enter the adjustment formula.

Case 2: Riester type adjustment ($b^{(1)} = 0.8$, $b^{(2)} = 0$)
With this specification the demographic change of the population continues to be irrelevant for the evolution of pension incomes and contributions. However, compared

to the previous case the reduction in $b^{(1)}$ leads to an increased sensitivity to changes in contribution rates such that this formula dampens the growth of pension payments. The formula corresponds to the one which was introduced for the German pension system as part of the so-called Riester reform in 2001.[5]

Case 3: Rürup type adjustment. ($b^{(1)} = 0.8$, $b^{(2)} = 0.5$)
This adjustment policy not only takes into account the previous increase in contribution rates but also the demographic change of the population. The parameter $b^{(2)}$ may be interpreted as a weight that shifts the demographic burden between retirees and workers. A larger value of $b^{(2)}$ reduces the growth rate of pensions thus shifting more of the burden to retired generations. In addition the increased sensitivity to changes in previous contribution rates is maintained by choosing $b^{(1)} < 1$. This corresponds to the adjustment formula which has been suggested by the so-called Rürup Kommission for the German pension system (see Rürup 2003). Both the reduction in $b^{(1)}$ and the increase in $b^{(2)}$ thus have a diminishing impact on pension incomes and dampen the growth in contribution rates.

Case 4: Gradual reduction. We compare the previous three adjustment policies with the gradual reduction of contributions as studied already in Section 5 where the initial contribution rate $\tau_t = 0.2$ is gradually lowered by 0.5% points in every period from $t = 51$ onwards such that $\tau_t = 0$ for $t \geq 90$.



(a) Net wage adjustment

(b) Riester type adjustment

(c) Rürup type adjustment

(d) Gradual reduction

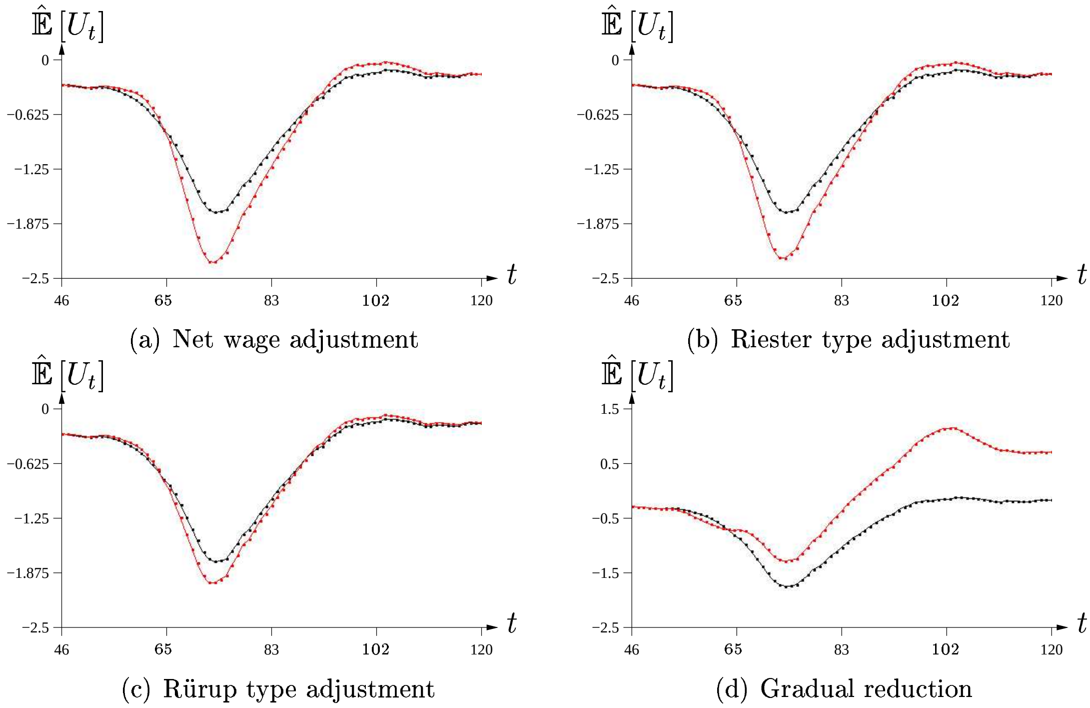
Figure 7: Time profiles of contribution rates under different adjustment policies

[5] In fact, in the actual Riester formula is slightly more complicated involving a gradual adjustment of the parameter $b^{(1)}$ over a time period of 10 years. Our specification corresponds to the long-run Riester formula after the year 2011, see Rürup (2003), p. 98.

The impact of either adjustment policy on the evolution of contribution rates is depicted in Figure 7. The initial contribution rate at time $t = 0$ has been set to $\tau_0 = 0.2$ and is represented by the black line in each of the figures. All parameters of the model as well as the underlying demographic scenario are the same as before.

In all four cases contributions remain more or less constant for $t \leq 50$, i.e., before the demographic transition occurs. In the first three cases the accelerating demographic change causes an increase in contribution rates from $t = 51$ onwards which become largest in Case 1 (up to $\approx 25.5\%$) and which is slightly dampened in Case 3 (up to $\approx 22.5\%$). After the demographic transition period contributions decrease again and eventually return to their initial value of 20% in all three cases. From this observation it is clear that all four policies under scrutiny may have an effect only during the demographic transition whereas the long run outcome will be exactly the same as that with a constant contribution rate studied in Section 2. Consequently we shall confine the following analysis to the welfare effects of the four reforms during the demographic transition period.

Figure 8 depicts the impact of either adjustment on consumer welfare. The black series represents the case where $\tau_t \equiv 0.2$ which serves as a reference case in all four scenarios.



(a) Net wage adjustment

(b) Riester type adjustment

(c) Rürup type adjustment

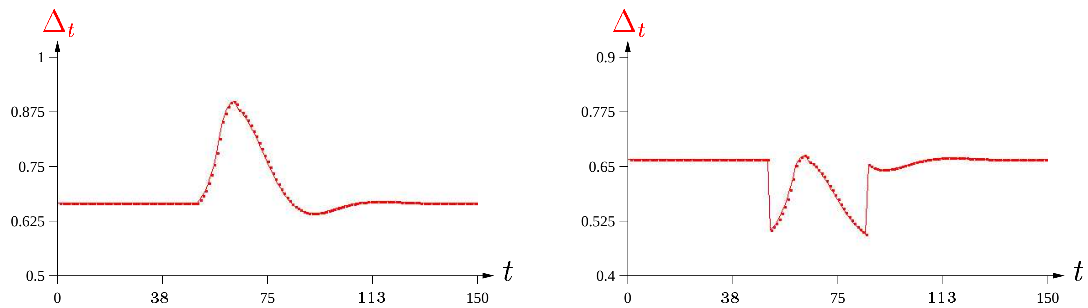
(d) Gradual reduction

Figure 8: Welfare effects under different adjustment policies

One observes that both the NWA adjustment as well as the Riester case yield almost identical results. The outcome in Figures 8(a) to 8(c) suggests that during the first phase of the demographic transition period ($t \leq 64$), it is still possible to stabilize pension incomes through an increase in contribution rates. Hence, these adjustment policies yield slightly higher utility during this time window as compared to the reference case where contributions remain at the initial 20% -level. However, this initial gain comes at a cost of a dramatic loss in the welfare of later generations. From $t = 65$ onwards, the utility level in both scenarios is significantly lower than in the reference case. More importantly, the increase in contribution rates leads to a much more dramatic downturn of utility during the time window $t \in \{65, \dots, 90\}$ when the demographic structure of the population is most unfavorable. Qualitatively Figure 8(c) conveys a similar impression as the previous one. The increase in contribution rates can attenuate the decrease in expected utilities only at the very beginning ($t \leq 64$). After this period, expected utility is lower than in the reference case and suffers a dramatic downturn. The dampened increase in contribution rates implied by the Rürup policy slightly attenuates this loss. Summarizing we find that any adjustment policy leading to a temporarily larger contribution rate fails to ameliorate and in fact even amplifies the welfare loss during the demographic transition. In contrast to that, Figure 8(d) shows that the gradual reduction in contribution rates slightly attenuates the downturn in expected utilities which occurs for $t \in \{65, \dots, 80\}$. Nevertheless, even a gradual reduction of contributions can not avoid a serious welfare loss during the demographic transition phase.

8 Increasing retirement age during transition

So far the analysis has focused on changes in contributions rates to adjust the pension system during demographic transitions. In this section, we consider a temporary increase in the retirement age as a another possible political measure to counteract the demographic problem. In this regard, recall that the retirement age is essentially determined by the parameter j_L which has been set to $j_L = 6$ during all the previous simulations. If we regard this parameter as being chosen by government authorities (rather than being determined e.g. by the physical capabilities of consumers) we can study the impact of a temporary decrease in j_L during the demographic transition phase. A huge advantage of this measure is that it is the only reform option which directly affects and decreases the dependency ratio by simultaneously increasing the number of workers and lowering the number of pensioners.

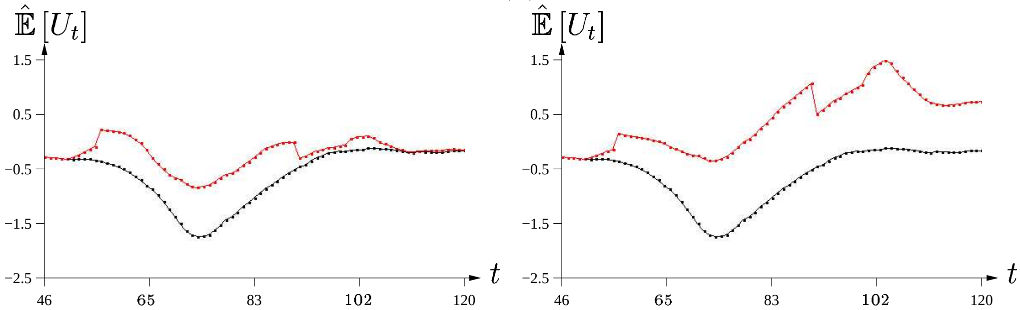


(a) Constant retirement age (b) Adjusted retirement age

Figure 9: Time profile of the dependency ratio

For the following experiments we shall assume the same demographic scenario as before and consider a temporary decrease to $j_L = 5$ during the time periods $t \in \{51, \dots, 85\}$. For $t < 51$ and $t > 85$ we have $j_L = 6$ as before. The impact of this policy on the structure of the population is depicted in Figure 9 showing the dependency ratio with and without a temporary adjustment of the retirement age.

One observes that the reduction in j_L leads to sudden fall in the dependency ratio in $t = 51$. In the subsequent periods, the series starts to increase again due to the accelerating demographic change. However, even in period $t = 63$ when the demographic effect reaches its maximum the dependency ratio only slightly exceeds its initial level of 66%. After this, it starts to decrease again since the population now gradually adjusts to its new long-run level. The increase in $t = 86$ back to $j_L = 6$ then shifts the ratio upwards to its initial (and new) long-run level.
Consider next the impact of the proposed adjustment on consumers' welfare during the demographic transition period. Assume first that contributions are not adjusted such that the contribution rate remains at its initial level, i.e., $\tau_t \equiv \tau = 0.2$. The evolution of expected utilities with (red series) and without (black) a temporary increase in retirement age is depicted in Figure 10(a). It shows that the increase in retirement



(a) $\tau = 0.2$ (b) $\tau_t = 0.2 \searrow 0$

Figure 10: Profiles of lifetime utilities with adjustments in retirement age

age leads to a jump in utility from $t = 56$ onwards. These are the generations which now have one additional period of labor income 'in exchange' for one period of pension income. At the same time the pension income increases because now there are fewer retirees who receive benefits from the pension system. This explains the gradual increase in utilities for $t \in \{51, \dots, 55\}$. From $t = 63$ onwards the demographic effect leads to a decrease in utility which nevertheless is much less serious than without an adjustment of the retirement age. In fact, compared to the reference case, the proposed adjustment leads to higher expected utility throughout the entire time window.
Consider now the case where the increase in the retirement age is combined with a gradual reduction of contribution rates as studied in the previous section such that the initial contribution rate $\tau_t = 20\%$ is gradually lowered by 0.5% points from $t = 51$ until $t = 85$ such that $\tau_t = 0$ for $t \geq 85$. In addition, the retirement age is adjusted during periods $t \in \{51, \dots, 85\}$ as before. Figure 10(b) compares the welfare effects of this combined policy to the reference case with $\tau \equiv 0.2$ and constant retirement age. The result seems very convincing: the combination of the two measures is capable of entirely eliminating the downturn induced by demographic change. Throughout the entire time

window expected utility never falls below its initial level but is in fact much larger in most periods. With these findings, it seems that a combination of a gradual decrease in contributions accompanied by a temporary increase in the retirement age is the most promising political measure to encounter the demographic problem.
Several remarks must be made at this point. Firstly, in our model, consumers do not derive disutility from labor, hence the additional period they have to work as the retirement age is increased does not have a diminishing impact on utility. Clearly, this may be debatable. If one takes into account the utility-diminishing effect of an increase in the retirement age, the result may change. A second point is that the utility levels in Figure 10(b) are still far from being equally distributed across the time window. This is to some extent due to our 'coarse' time scale with one time unit corresponding to four years. An adjustment of the retirement age on an annual basis could lead to an even smoother adjustment of utilities. This could be accompanied by an improved adjustment policy where contributions are not necessarily decreased in constant steps but the decrease changes in each period. These two refinements should lead to an even smoother distribution of utility over the respective generations.

9 Conclusions

This paper has studied the welfare implications of pension systems from two different perspectives. The first view dealt with the long-run welfare effects corresponding to the case with a stationary population. In this regard, the explicit modeling approach put forward in this paper in conjunction with random dynamical systems theory was used to develop a concept which measures the long-run welfare implications of alternative pension systems by comparing the welfare along the corresponding stationary solution of the system. The application of this concept strongly suggest the long-run inefficiency of a public pension system where any reduction of contributions leads to a welfare improvement.

The second perspective allows to study the performance of alternative pension policies describing adjustments of the system during demographic transition. A demographic scenario has been developed which roughly matches the predicted evolution of the German population over the next fifty years. It has been shown that without substantial reforms the predicted demographic change of the population will lead to large welfare losses on the part of consumers. Among the reform scenarios discussed, a temporary increase in the retirement age accompanied by a gradual reduction in contributions was shown to be most promising to overcome the demographic losses.

A Mathematical appendix

A.1 Concepts from random dynamical systems

To embed the dynamics of the model into the framework of random dynamical systems theory, assume as in Section 2 that the contribution rate to the pension system is constant such that $\tau_t \equiv \tau$. For each t let ξ_t be defined as in (25) and let the map $\phi_\tau\big(\cdot;\eta_t\big)$ be defined as in Section 2 with suitable domain $\Xi \subset \mathbb{R}_{++}^{J+1} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}^J \times \mathbb{R}_+^J$.

For the equation $\xi_t = \phi_\tau(\xi_{t-1},\eta_t)$ to define a random dynamical system we need to write the noise process $\{\eta_t\}_{t\in\mathbb{Z}}$ characterized in Assumption 1 as a so-called metric dynamical system $(\Omega,\mathcal{F},\mathbb{P},(\vartheta^t)_{t\in\mathbb{Z}})$ which constitutes the first building block of a random dynamical system. For this purpose, endow the product space $\Omega := \prod_{t\in\mathbb{Z}}[0,\eta_{max}]$ with its Borel-σ algebra $\mathcal{F} := \mathcal{B}(\Omega)$ and the product measure $\mathbb{P} := \otimes_{t\in\mathbb{Z}}\nu_\eta$. The left shift on Ω is defined as the map $\vartheta : \Omega \longrightarrow \Omega$, $(\tilde\omega_t)_{t\in\mathbb{Z}} \longmapsto \vartheta(\tilde\omega_t)_{t\in\mathbb{Z}} := (\tilde\omega_{t+1})_{t\in\mathbb{Z}}$ the inverse of which is denoted as ϑ^{-1}. For each $t\in\mathbb{Z}$ let ϑ^t denote the tth iterate of the map ϑ if $t>0$ and of ϑ^{-1} if $t<0$, respectively. With the help of the evaluation map $\eta : \Omega \longrightarrow [0,\eta_{max}]$, $\eta(\tilde\omega_t)_{t\in\mathbb{Z}} := \tilde\omega_0$ the original process $\{\eta_t\}_{t\in\mathbb{Z}}$ can be written in the form $\{\eta(\vartheta^t)\}_{t\in\mathbb{Z}}$, i.e., for each $t\in\mathbb{Z}$ and $\tilde\omega\in\Omega$, $\eta_t(\tilde\omega) = \eta(\vartheta^t\tilde\omega)$. Note that by construction the measure $\mathbb{P}$ is invariant with respect to ϑ, i.e. for all $A\in\mathcal{F}$ one has $\mathbb{P}(\vartheta A) = \mathbb{P}(A)$. In addition we shall assume that $\mathbb{P}$ is ergodic with respect to ϑ, i.e. for each $A\in\mathcal{F}$ which is invariant under the map ϑ (i.e. $\vartheta A = A$) one has $\mathbb{P}(A)\in\{0,1\}$. The quadruple $(\Omega,\mathcal{F},\mathbb{P},(\vartheta^t)_{t\in\mathbb{Z}})$ defines an ergodic metric dynamical system in the sense of Arnold (1998) which constitutes the first building block of a random dynamical system.

It follows that for each fixed $\tilde\omega\in\Omega$ and initial state $\xi_0\in\Xi$ the evolution of the system can be written as $\xi_t = \phi_\tau\big(\xi_{t-1};\eta(\vartheta^t\tilde\omega)\big)$, $t\geq 0$. To alleviate the notation write $\phi_\tau(\eta(\tilde\omega)) := \phi_\tau(\cdot;\eta(\tilde\omega)) : \Xi \longrightarrow \Xi$, $\xi \longmapsto \phi_\tau(\eta(\tilde\omega))\,\xi := \phi_\tau(\xi;\eta(\tilde\omega))$ for each $\tilde\omega\in\Omega$ and let $\mathbb{T} := \mathbb{N}_0$. The iteration of the map $\phi_\tau(\cdot)$ defines a measurable flow $\Phi_\tau : \mathbb{T}\times\Omega\times\Xi \longrightarrow \Xi$

$$\Phi_\tau(t,\tilde\omega,\xi_0) := \begin{cases} \xi_0 & t=0 \\ \phi_\tau(\eta(\vartheta^t\tilde\omega))\circ\ldots\circ\phi_\tau(\eta(\vartheta\tilde\omega))\;\xi_0 & t\geq 1 \end{cases} \tag{51}$$

Equation (51) together with the metric dynamical system $(\Omega,\mathcal{F},\mathbb{P},(\vartheta^t)_{t\in\mathbb{Z}})$ defines a random dynamical system in the sense of Arnold (1998) with one-sided time. The value $\Phi_\tau(t,\tilde\omega,\xi_0)\in\Xi$ determines the state of the system at time $t\in\mathbb{T}$ from the initial state $\xi_0\in\Xi$ and the path $\tilde\omega\in\Omega$ of the perturbation.

The following definition introduces the concept of a random fixed point which is the stochastic analogue to a fixed point of a deterministic dynamical system and which plays a crucial role in what follows.

Definition 2

Let $\tau\in[0,\bar\tau]$ be fixed. A random fixed point of the random dynamical system defined by (51) is a random variable $\xi_\tau^\star : \Omega \longrightarrow \Xi$ with the property that for each $\tilde\omega\in\Omega$

$$\xi_\tau^\star(\vartheta\tilde\omega) = \phi_\tau(\eta(\vartheta\tilde\omega),\xi_\tau^\star(\tilde\omega)) = \phi_\tau(\eta(\vartheta\tilde\omega))\xi_\tau^\star(\tilde\omega). \tag{52}$$

A random fixed point $\xi^\star_\tau$ is said to be asymptotically stable if for each fixed $\tilde{\omega} \in \Omega$ there exists a random set $U(\tilde{\omega}) \subset \Xi$ such that

$$\lim_{t\to\infty} \|\xi^\star_\tau(\vartheta^t\tilde{\omega}) - \Phi_\tau(t,\tilde{\omega},\xi)\| = 0 \tag{53}$$

for all $\xi \in U(\tilde{\omega})$ $\mathbb{P}$-a.s.

In the stable case the path $t \longmapsto \Phi_\tau(t,\tilde{\omega},\xi_0)$ will asymptotically move as the corresponding path $t \longmapsto \xi^\star_\tau(\vartheta^t\tilde{\omega})$ of the random fixed point ($\mathbb{P}$-a.s.). Clearly, for those components of the map ϕ_τ which are deterministic (and thus independent of $\tilde{\omega} \in \Omega$), the property (52) coincides with that of a deterministic fixed point.

A.2 Proof of Lemma 2

For all deterministic components of $\xi^\star$ the definition (52) coincides with that of a deterministic fixed point and is therefore implied by Lemma 1 and by asymptotic stability as required in Definition 1 (i). Stability of the cum-dividend price process is an immediate consequence of Definition 1 (ii).

A.3 Proof of Theorem 1

As shown in Hillebrand (2007) for each τ the values $\mu^\star_\tau$ and $R^\star_\tau$ are related to $K^\star_\tau$ as

$$\begin{aligned} \mu^\star_\tau &= \frac{\bar{\eta}}{\bar{x}} + \frac{\kappa\,\bar{N}^\alpha K^{\star\,1-\alpha}_\tau}{\bar{x}}\left[\frac{\alpha\,\beta}{1+\beta}(1-\tau) + (1-\alpha)(1-\frac{1}{\gamma_1})\right] \\ &\quad -\frac{g'(\delta)}{\bar{x}}\frac{K^\star_\tau}{1-\alpha}\left[\frac{\alpha\,\tau}{1+\beta} + \frac{1-\alpha}{\gamma_1}\right] > \frac{\bar{\eta}}{\bar{x}} \qquad (54) \\ R^\star_\tau &= \frac{(1-\alpha)\kappa}{g'(\delta)}\left(\frac{\bar{N}}{K^\star_\tau}\right)^\alpha > 1. \qquad (55) \end{aligned}$$

Here g' denotes the first derivative of the map g defined in (10). Since each of the mappings is differentiable, it follows that the derivatives of (54) and (55) take the form

$$\begin{aligned} \partial_\tau\mu^\star_\tau &= -\frac{\kappa\,\bar{N}^\alpha K^{\star\,1-\alpha}_\tau}{\bar{x}}\left[\frac{\alpha}{1+\beta}\Big(\beta + R^{\star-1}_\tau\Big) - (1-\alpha)\frac{\partial_\tau K^\star_\tau}{K^\star_\tau}\left(\frac{\alpha\beta}{1+\beta}(1-\tau)\right.\right. \qquad (56) \\ &\quad \left.\left. +(1-\alpha)(1-\gamma_1^{-1}) - R^{\star-1}_\tau\Big(\frac{\alpha}{1-\alpha}\frac{\tau}{1+\beta} + \gamma_1^{-1}\Big)\right)\right] < 0 \\ \partial_\tau R^\star_\tau &= -\alpha R^\star_\tau\frac{\partial_\tau K^\star_\tau}{K^\star_\tau} > 0. \qquad (57) \end{aligned}$$

Differentiability of the maps $K^\star, R^\star$ and $\mu^\star$ together with the properties of the distribution $\nu_{\tilde{\eta}}$ imply that the map $\tau \longmapsto \mathbb{E}[U^\star_\tau]$ in (40) is differentiable and differentiation and integration may be interchanged (cf. Bauer (1992)). Using (57) the derivative computes

$$\begin{aligned}\partial_\tau \mathbb{E}[U_\tau^\star] &= -\frac{R_\tau^\star - 1}{\tau + (1-\tau)R_\tau^\star} + \left[1 - \alpha + \frac{\alpha\tau}{\tau + (1-\tau)R_\tau^\star}\right]\frac{\partial_\tau K_\tau^\star}{K_\tau^\star} \\ &\quad + \beta \int_{[-\bar\eta,\bar\eta]} \frac{\alpha\kappa \bar N^\alpha {K_\tau^\star}^{1-\alpha} + \bar x\, \partial_\tau \mu_\tau^\star + \partial_\tau K_\tau^\star (1-\alpha)\kappa \bar N^\alpha {K_\tau^\star}^{-\alpha}\big(\alpha\tau + \frac{1-\alpha}{\gamma_1}\big)}{\kappa \bar N^\alpha {K_\tau^\star}^{1-\alpha}\big(\alpha\tau + \frac{1-\alpha}{\gamma_1}\big) + \bar x\, \mu_\tau^\star + \tilde\eta} d\tilde\eta. \end{aligned} \tag{58}$$

Let $\tau \in [0,1[$ be arbitrary but fixed. We want to show that $\partial_\tau \mathbb{E}[U_\tau^\star] < 0$. Note that the first two terms in (58) are always negative due to (55) and (57). In addition, the denominator of the integrand in (58) is strictly positive for all $\tilde\eta \in [-\bar\eta, \bar\eta]$ due to (54) implying that the integral in (58) satisfies

$$0 < \int_{[-\bar\eta,\bar\eta]} \frac{d\tilde\eta}{\kappa \bar N^\alpha {K_\tau^\star}^{1-\alpha}\big(\alpha\tau + \frac{1-\alpha}{\gamma_1}\big) + \bar x\, \mu_\tau^\star + \tilde\eta} < \frac{1}{\kappa \bar N^\alpha {K_\tau^\star}^{1-\alpha}\big(\alpha\tau + \frac{1-\alpha}{\gamma_1}\big) + \bar x\, \mu_\tau^\star - \bar\eta}. \tag{59}$$

It follows from (59) that if the numerator in the integrand in (58) is non-positive, the claim is automatically satisfied. If the numerator is positive, equation (59) implies that

$$\begin{aligned}\partial_\tau \mathbb{E}[U_\tau^\star] &< -\frac{R_\tau^\star - 1}{\tau + (1-\tau)R_\tau^\star} + \left[1 - \alpha + \frac{\alpha\tau}{\tau + (1-\tau)R_\tau^\star}\right]\frac{\partial_\tau K_\tau^\star}{K_\tau^\star} \\ &\quad + \beta \frac{\alpha\kappa \bar N^\alpha {K_\tau^\star}^{1-\alpha} + \bar x\, \partial_\tau \mu_\tau^\star + \partial_\tau K_\tau^\star (1-\alpha)\kappa \bar N^\alpha {K_\tau^\star}^{-\alpha}\big(\alpha\tau + \frac{1-\alpha}{\gamma_1}\big)}{\kappa \bar N^\alpha {K_\tau^\star}^{1-\alpha}\big(\alpha\tau + \frac{1-\alpha}{\gamma_1}\big) + \bar x\, \mu_\tau^\star - \bar\eta} =: \bar U_\tau'. \end{aligned} \tag{60}$$

It suffices to show that $\bar U_\tau' < 0$. Using (54), (55) and (56) in (60) gives:

$$\begin{aligned}\bar U_\tau' &= -\frac{R_\tau^\star - 1}{\tau + (1-\tau)R_\tau^\star} + \left[\frac{\tau + (1-\alpha)(1-\tau)R_\tau^\star}{\tau + (1-\tau)R_\tau^\star}\right]\frac{\partial_\tau K_\tau^\star}{K_\tau^\star} \\ &\quad + \beta \frac{\frac{\alpha}{1+\beta}\Big(1 - {R_\tau^\star}^{-1}\Big) + (1-\alpha)\frac{\partial_\tau K_\tau^\star}{K_\tau^\star}\Big[\frac{\alpha\tau}{1+\beta} + \frac{\alpha\beta}{1+\beta} + 1 - \alpha - {R_\tau^\star}^{-1}\big(\frac{\alpha}{1-\alpha}\frac{\tau}{1+\beta} + \gamma_1^{-1}\big)\Big]}{\frac{\alpha\beta}{1+\beta} + \frac{\alpha\tau}{1+\beta}(1 - {R_\tau^\star}^{-1}) + (1-\alpha)(1 - \gamma_1^{-1}{R_\tau^\star}^{-1})} \qquad (61) \\ &= -\frac{R_\tau^\star - 1}{\tau + (1-\tau)R_\tau^\star} + \frac{R_\tau^\star - 1}{R_\tau^\star + \frac{\tau}{\beta}(R_\tau^\star - 1) + \frac{1-\alpha}{\alpha}\frac{1+\beta}{\beta}(R_\tau^\star - \gamma_1^{-1})} \qquad (62) \\ &\quad + \frac{\partial_\tau K_\tau^\star}{K_\tau^\star}\left[\frac{\tau + (1-\alpha)(1-\tau)R_\tau^\star}{\tau + (1-\tau)R_\tau^\star} + (1-\alpha)\frac{R_\tau^\star(\beta+\tau) + \frac{1-\alpha}{\alpha}(1+\beta)R_\tau^\star - \frac{\tau}{1-\alpha} - \frac{1+\beta}{\alpha\gamma_1}}{R_\tau^\star + \frac{\tau}{\beta}(R_\tau^\star - 1) + \frac{1-\alpha}{\alpha}\frac{1+\beta}{\beta}(R_\tau^\star - \gamma_1^{-1})}\right]\end{aligned}$$

Since $0 < \tau + (1-\tau)R_\tau^\star < R_\tau^\star < R_\tau^\star + \frac{\tau}{\beta}(R_\tau^\star - 1) + \frac{1-\alpha}{\alpha}\frac{1+\beta}{\beta}(R_\tau^\star - \gamma_1^{-1})$ the first line in (62) is negative. We show that the term in brackets in the second line is positive. By the previous observation it suffices to show

$$\tau + (1-\alpha)(1-\tau)R_\tau^\star > -(1-\alpha)\Big(R_\tau^\star(\beta+\tau) + \frac{1-\alpha}{\alpha}(1+\beta)R_\tau^\star - \frac{\tau}{1-\alpha} - \frac{1+\beta}{\alpha\gamma_1}\Big)$$

which is equivalent to

$$(1+\beta)R_\tau^\star > \frac{1+\beta}{\alpha\gamma_1} - \frac{1-\alpha}{\alpha}(1+\beta)R_\tau^\star$$

and which follows immediately from $R_\tau^\star > 1 > \gamma_1^{-1}$ completing the proof. ■

A.4 Proof of Lemma 4

Since any steady state $(\bar{K}, \bar{R}) \gg 0$ of the capital dynamics (27) has to satisfy

$$\bar{K} = \left[\frac{\kappa(1-\alpha)}{g(\delta)} \frac{\bar{N}^{\alpha}}{\gamma_1 \bar{R}}\right]^{\frac{1}{\alpha}}$$

the claim is an immediate consequence of equation (43). ■

A.5 Proof of Theorem 2

Exploiting (55) and the fact that $R^{\star}_{\tau} > 1 > \frac{1}{\gamma_1}$ for each τ together with the functional form (10) of g and recalling that $\bar{L}^S = \bar{N}$ gives

$$K^{\star}_{\tau} = \left[\frac{\kappa(1-\alpha)}{g'(\delta)} \frac{\bar{N}^{\alpha}}{R^{\star}_{\tau}}\right]^{\frac{1}{\alpha}} = \left[\frac{\kappa(1-\alpha)}{g(\delta)} \frac{\bar{N}^{\alpha}}{\gamma_1 R^{\star}_{\tau}}\right]^{\frac{1}{\alpha}} < \left[\frac{\kappa(1-\alpha)}{g(\delta)} \bar{L}^{S\,\alpha}\right]^{\frac{1}{\alpha}} = K^{opt}.$$

■

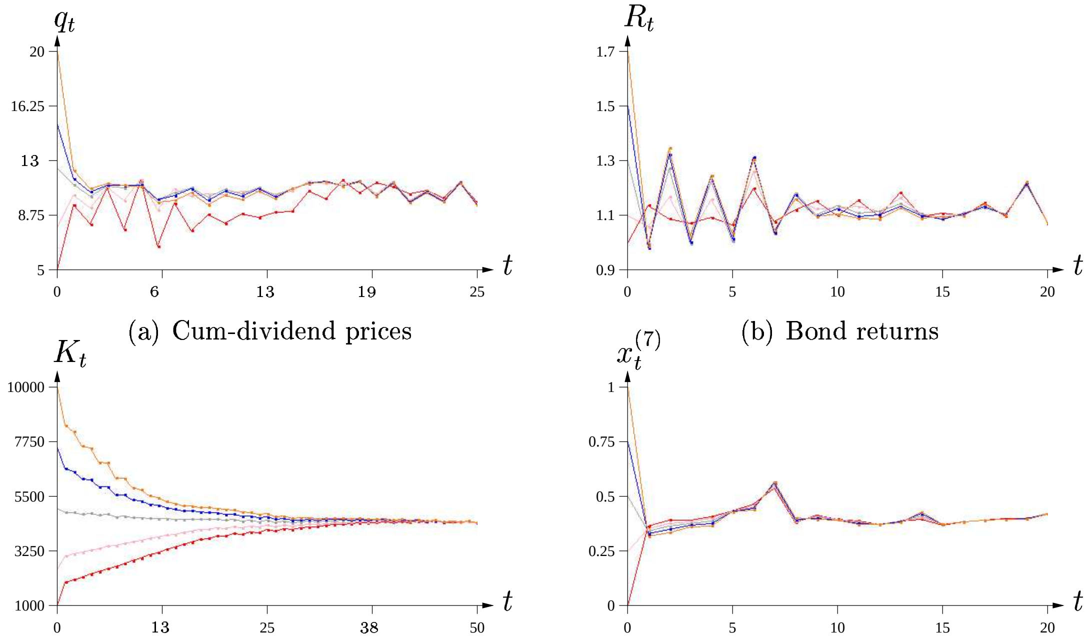
A.6 Proof of Lemma 5

Utilizing the definitions from (45) and (48) and exploiting the stability property (53) the statement in (i) follows immediately from the continuity of the involved mappings and the triangle inequality. The second statement is then a consequence of the continuity of the logarithm and the triangle inequality. ■

A.7 Stability of random fixed points

The welfare concept developed in Sections 3 and 4 requires existence of a stable a random fixed point describing the long-run stationary evolution of the random dynamical system (33). While Lemma 2 and 3 together ensure existence for the deterministic two-period case, a general theoretical result for the stochastic multi-period case is not available yet. For this reason this section provides numerical evidence that for the parameter set listed in Table 1 a stable random fixed point exists. We show that the long-run behavior of the model is stationary and does not depend on the initial state $\xi_0 \in \Xi$. Figure 11 depicts time series of selected state variables for five different initial values. These are the time series of cum-dividend prices q_t, bond returns R_t, capital stock K_t and the share portfolios $x^{(7)}_t$ of generation $j = 7$. Qualitatively, the result is the same for all other components of the state vector ξ_t defined in (25). Note that the realization of the noise process is the same in each case. The contribution rate is $\tau = 0.1$, however, the qualitative result is the same for each $\tau \in [0, 0.2]$.

In either case the respective state variable converges to a unique stationary sample path, independently of its initial value. Moreover, in either case, convergence occurs within



(a) Cum-dividend prices

(b) Bond returns

(c) Capital stock

(d) Stock portfolios of generation $j = 7$

Variable:	Value 1	Value 2	Value 3	Value 4	Value 5
q_0	8	10	12	14	16
R_0	1.0	1.1	1.2	1.3	1.4
K_0	1000	5000	10000	15000	20000
$x_0^{(7)}$	0	0.25	0.5	0.75	1

Figure 11: Convergence of selected state variables for different initial values, $\tau = 0.1$

the first 50 periods and is in fact even much faster for the financial variables q_t and R_t. It can be shown that the same result holds true for all other state variables contained in the state vector ξ_t of the model. These results strongly support the existence of a unique sample path governs the long-run behavior of the model in the sense that all sample paths pertaining to different initial conditions eventually behave like the corresponding path of the random fixed point.

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