Uniqueness of Markov Equilibrium in Stochastic OLG Models with Nonclassical Production

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December 6, 2013

Abstract
This paper studies Markov Equilibria (ME) corresponding to recursive equilibria on the natural state space in the stochastic OLG model extended to include non-additive utility, nonclassical production, and Markovian production shocks. Specifically, we provide sufficient conditions under which the ME is unique. It turns out that uniqueness obtains for a large class of economies and that restrictions either on the consumption side or the production side alone are sufficient to guarantee this result. We also discuss additional properties such as continuity or smoothness of the equilibrium mappings and whether additional recursive or non-recursive equilibria exist.

JEL classification: C62, D51, E32.
Keywords: Markov equilibrium, Uniqueness, Overlapping generations, Nonclassical production, Markovian production shocks
Introduction

Starting with the pioneering work of Wang (1993), researchers have studied the existence and properties of equilibria in overlapping generations (OLG) models with random production shocks. Of particular interest in these studies is the class of recursive equilibria (RE) where the equilibrium variables are determined by time-invariant mappings on the natural state space with the state variable consisting of current capital and the production shock. Following the terminology introduced in Kübler & Polemarchakis (2004), such equilibria will be referred to as Markov equilibria (ME). Studying the properties of ME for a large class of stochastic OLG models is the theme of the present paper.

In a setting with classical production functions, i.i.d. shocks, and time-additive utility, Wang (1993) showed that a capital-income monotonicity condition on the production technology is sufficient for a unique ME to exist. In addition, he established several additional properties of the equilibrium mappings such as smoothness and monotonicity. The model in Wang (1993) was generalized in Wang (1994) to include non-additive utility and general Markovian shocks processes and in McGovern, Morand & Reffett (2013) and Morand & Reffett (2007) who, in addition, allow for non-classical production functions. The latter were originally introduced in Greenwood & Huffman (1995) and Coleman (1991) in a non-OLG setting. While Wang (1994) studies the existence of so-called Generalized Markov Equilibria (GME) where the state space may include additional variables such as sunspots, etc., Morand & Reffett (2007) continue to focus on ME. Using methods from functional analysis, they develop a general approach to study ME as fixed points of a suitably defined operator. This approach permits them to derive general sufficient conditions for a ME to exist in their setup.

Building on the existence results of Morand & Reffett (2007), the present paper seeks to establish additional properties of ME while maintaining the same level of generality as their study. As our main contribution, we establish sufficient conditions under which the ME is unique and the equilibrium mappings possess additional properties such as monotonicity, continuity, or even smoothness. Similar properties were derived in Wang (1993) but it is not yet known under what conditions they hold for the much larger class of economies studied in Morand & Reffett (2007). Knowing these additional properties, however, seems important not only for theoretical reasons and welfare analysis, but also for applied numerical work.

Conceptually, the present paper employs the same operator-based approach used in Morand & Reffett (2007). In general, uniqueness of a fixed point obtains only under very special circumstances, e.g., if the underlying operator is a contraction or satisfies a set of additional and rather restrictive conditions as in Coleman (1991). However, we demonstrate in this paper that the operator developed in Morand & Reffett (2007) possesses a very special structure that is unique to their OLG setup with two-period lived consumers but not exploited in their paper. It is precisely this additional property that will allow us to obtain the uniqueness results of this paper. In fact, it will turn out that the ME is unique under the restrictions imposed in Morand & Reffett (2007). Moreover, we also show that if the ME is unique, it is in fact the unique sequential equilibrium of the economy.

The paper is organized as follows. Section 1 introduces the model. The formal structure to study ME is developed in Section 2. The main results are presented in Section 3. Section 4 concludes, proofs for all results are placed in the appendix.
1 The Model

This section presents the basic setup of the model which extends the one in Morand & Reffett (2007) by relaxing several of their assumptions.

Production sector
The production sector produces a consumption good using labor and capital as inputs. In addition, production in period $t$ is subjected to a random shock $\varepsilon_t$ with values in the compact set $E \subset [\varepsilon_{\text{min}}, \varepsilon_{\text{max}}]$. In equilibrium, labor is constant and the wage $w_t$ and capital return $r_t$ are determined from current capital $k_t > 0$ and the shock $\varepsilon_t \in E$ by the maps

\[
\mathcal{W} : \mathbb{R}_{++} \times E \rightarrow \mathbb{R}_{++}, \quad w_t = \mathcal{W}(k_t, \varepsilon_t) \tag{1a}
\]
\[
\mathcal{R} : \mathbb{R}_{++} \times E \rightarrow \mathbb{R}_{++}, \quad r_t = \mathcal{R}(k_t, \varepsilon_t). \tag{1b}
\]

Profits are zero at equilibrium. The previous specification includes the cases with classical production in Wang (1993) and non-classical production in Greenwood & Huffman (1995), Coleman (1991), and Morand & Reffett (2007) as special cases. Rather than specifying the underlying production technology directly, we will work with the mappings $\mathcal{W}$ and $\mathcal{R}$ as being part of the primitives of the economy. The following restrictions are imposed which are slightly weaker than those in Morand & Reffett (2007).

Assumption 1
(i) Both functions $\mathcal{W}$ and $\mathcal{R}$ are Borel-measurable.

(ii) For each $\varepsilon \in E$, $k \mapsto \mathcal{W}(k, \varepsilon)$ is increasing while $k \mapsto \mathcal{R}(k, \varepsilon)$ is strictly decreasing.

(iii) For each $k > 0$, $\varepsilon \mapsto \mathcal{W}(k, \varepsilon)$ and $\varepsilon \mapsto \mathcal{R}(k, \varepsilon)$ are bounded.

Production shocks follow a Markov process with time-invariant transition probability $Q : E \times \mathcal{B}(E) \rightarrow [0, 1]$. Given an initial state $\varepsilon_0 \in E$, the transition $Q$ permits to construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ to which the process $\{\varepsilon_t\}_{t \geq 0}$ is adapted such that $\varepsilon_t : \Omega \rightarrow E$ is $\mathcal{F}_t$-measurable, $t \geq 0$. We denote by $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_t]$ the expectation conditional on the information represented by $\mathcal{F}_t$. Being a transition probability, $Q$ preserves measurability, i.e., if $f : Y \times E \rightarrow \mathbb{R}$, $Y \subset \mathbb{R}^m$, $m \geq 1$ is a measurable function, so is $g : Y \times E \rightarrow \mathbb{R}$, $g(y, \varepsilon) := \int_E f(y, \varepsilon')Q(\varepsilon, d\varepsilon')$. A stronger requirement frequently imposed would be the so-called Feller property: If $Y \subset \mathbb{R}^m$, $m \geq 1$ and $f : Y \times E \rightarrow \mathbb{R}$ is a bounded continuous function, then so is $g : Y \times E \rightarrow \mathbb{R}$, $g(y, \varepsilon) := \int_E f(y, \varepsilon')Q(\varepsilon, d\varepsilon')$. We will refrain from assuming this property except in Theorem 2 (iii).

Consumption sector
The consumption sector consist of overlapping generations of consumers who live for two periods. A young consumer in period $t \geq 0$ supplies one unit of labor inelastically to receive the wage $w_t > 0$ which is used for saving $s_t$ and youthful consumption $c_t^y = w_t - s_t$. Savings earn the random capital return $r_{t+1}$ in the following period in which no further income is received. Thus, old-age consumption is given by the random variable $c_{t+1}^o = s_tr_{t+1}$. Given labor income $w_t > 0$ and the perceived random capital return of the following period $r_{t+1}$,
consumers choose savings $s_t$ to maximize expected lifetime utility based on some von-Neumann Morgenstern utility function $U: \mathbb{R}_+^2 \rightarrow \mathbb{R}$ defined over consumption in both periods. The decision problem reads

$$\max_s \left\{ \mathbb{E}_t [U(w_t - s, sr_{t+1})] \mid 0 \leq s \leq w_t \right\}. \quad (2)$$

The following restrictions are imposed on $U$.

**Assumption 2**

$U$ is $C^2$, strictly concave with derivatives satisfying $U_{ii} < 0$, $i \in \{1, 2\}$ and the Inada conditions $\lim_{c^y \to 0} U_1(c^y, c^o) = \infty$ for all $c^o > 0$ and $\lim_{c^o \to 0} U_2(c^y, c^o) = \infty$ for all $c^y > 0$.

Given $w_t > 0$ and the random variable $r_{t+1}$, Assumption 2 guarantees a unique interior solution $0 < s_t < w_t$ to (2) which determines next period’s capital stock $k_{t+1}$.

**Equilibrium**

The economy is $\mathcal{E} = (W, R, Q, U)$ plus initial condition $x_0 := (k_0, \varepsilon_0) \in X := \mathbb{R}_{++} \times \mathcal{E}$. The set $X$ will be referred to as the (natural) state space of the economy. The following definition provides a general characterization of equilibrium. Note that the previous assumptions imply that all equilibrium variables are strictly positive.

**Definition 1**

Given $x_0 \in X$, a sequential equilibrium (SE) of $\mathcal{E}$ is an adapted process $\{w_t, r_t, s_t, k_{t+1}\}_{t \geq 0}$ with values in $\mathbb{R}_{++}^4$ satisfying the following conditions for all $t \geq 0$:

(i) Given $w_t$ and the random variable $r_{t+1}$, $s_t$ solves (2) while $k_{t+1} = s_t$.

(ii) Factor prices $w_t$ and $r_t$ are determined from $k_t$ and $\varepsilon_t$ by (1a) and (1b).

The induced equilibrium consumption processes $\{c^y_t, c^o_t\}_{t \geq 0}$ follow directly by inserting the equilibrium variables into the consumers’ budget constraints.

## 2 Markov Equilibria (ME)

**Definition of ME**

A recursive equilibrium (RE) is an equilibrium where all variables of period $t$ are determined by time-invariant functions of a state variable with values in a suitable state space. A Markov equilibrium (ME) is a RE on the natural state space $X = \mathbb{R}_{++} \times \mathcal{E}$ where the state variable is $x_t := (k_t, \varepsilon_t)$. It is this class of equilibria that we will focus on in this paper. As the functions $W$ and $R$ already satisfy the Markov property, a ME is essentially determined by a time-invariant mapping which determines the evolution of capital respectively savings. Formally, we have

**Definition 2**

Given $x_0 \in X$, a ME of $\mathcal{E}$ is a measurable map $\mathcal{K}: X \rightarrow \mathbb{R}_{++}$ on the natural state space $X$ such that the process $\{w_t, r_t, s_t, k_{t+1}\}_{t \geq 0}$ defined recursively as $k_{t+1} = \mathcal{K}(k_t, \varepsilon_t) = s_t$, $w_t = W(k_t, \varepsilon_t)$, and $r_t = R(k_t, \varepsilon_t)$ for all $t \geq 0$ is a SE of $\mathcal{E}$.
Constructing the operator $A$

To establish the existence and properties of ME, we follow Morand & Reffett (2007) to construct an operator $A$ on a suitable function space $\mathcal{S}$ whose fixed points are ME. In the sequel we take $\mathcal{S}$ to be the class of Borel-measurable functions $\mathcal{K}: \mathbb{X} \rightarrow \mathbb{R}^{++}$. The operator $A$ is constructed from the Euler equations derived from the consumer’s decision problem \[2\]. To this end, consider a given period $t$ with state $x_t = (k_t, \varepsilon_t)$ which determines the current wage $w_t$ from \[1a\] and the conditional distribution $Q(\varepsilon_{t+1} \mid \cdot)$ of next period’s shock. To decide on her investment $s_t \in [0, \omega]$, the consumer needs to determine the (correct) distribution of the uncertain capital return $r_{t+1}$ of the following period. As $r_{t+1} = R(k_{t+1}, \varepsilon_{t+1})$ and the consumer knows the function $R$ and the conditional distribution of $\varepsilon_{t+1}$ (which are part of the fundamentals of the economy), this amounts to (correctly) forecasting next period’s capital stock $k_{t+1} > 0$ which, conditional on the information at time $t$, is a value rather than a random variable. Suppose the consumer holds a perceived law of motion for the capital stock $K \in \mathcal{S}$ to compute her forecast $k_{t+1} = K(k_t, \varepsilon_t)$. Then, given $(x_t, k_{t+1}) \in \mathbb{X} \times \mathbb{R}^{++}$, an optimal savings decision $s_t$ derived from \[2\] must satisfy the first-order conditions $H(s_t; k_t, \varepsilon_t, k_{t+1}) = 0$ where

\begin{equation}
H(s; k, \varepsilon, k') := -\int_{\mathcal{E}} U_1(W(k, \varepsilon) - s, sR(k', \varepsilon'))Q(\varepsilon, d\varepsilon') + \int_{\mathcal{E}} R(k', \varepsilon')U_2(W(k, \varepsilon) - s, sR(k', \varepsilon'))Q(\varepsilon, d\varepsilon').
\end{equation}

Under the Inada assumptions (which ensure existence) and the strict concavity of $U$ (which implies uniqueness), the function $H(\cdot; x, k')$ has a unique zero for all $x \in \mathbb{X}$ and $k' > 0$. Thus, there exists a (savings) function $S: \mathbb{X} \times \mathbb{R}^{++} \rightarrow \mathbb{R}^{++}$ which determines the unique solution $s_t = S(k_t, \varepsilon_t, k_{t+1})$ to \[2\]. The following lemma is the key ingredient to define the operator $A$ below.

\textbf{Lemma 2.1}

Under Assumptions \[1\] and \[2\] the mapping $S: \mathbb{X} \times \mathbb{R}^{++} \rightarrow \mathbb{R}^{++}$ is measurable.

Substituting the perceived law of motion $K$ into $S$ one obtains an operator $A$ which associates with $K \in \mathcal{S}$ the new function $AK: \mathbb{X} \rightarrow \mathbb{R}^{++}$ defined as

\begin{equation}
(\mathcal{A}K)(x) := S(x, K(x)) \quad \text{for} \quad x \in \mathbb{X}.
\end{equation}

As $S$ is measurable by Lemma \[2.1\] and the composition of measurable functions is again measurable, it is evident that $A$ maps $\mathcal{S}$ into itself. As $k_{t+1} = s_t$ at equilibrium, a ME corresponds precisely to a fixed point of $A$.

Fixed points of $A$

Morand & Reffett (2007) establish the existence of fixed points of $A$ by imposing additional restrictions such as monotonicity on the set $\mathcal{S}$. In this paper, we follow a different route by determining fixed points of $A$ pointwise as zeroes of a real-valued function. This approach is possible due to a key property of the operator $A$ which is evident from \[1\] but not exploited in Morand & Reffett (2007): For each $x \in \mathbb{X}$, the value $(\mathcal{A}K)(x)$ depends only on the value $K(x)$.
$K(x)$ and not on the entire function $\mathcal{K}$. Formally, for any two functions $K_1, K_2 \in \mathcal{S}$, $K_1(x) = K_2(x)$ implies $(AK_1)(x) = (AK_2)(x)$. The main implication is that fixed-points of $A$ can be constructed pointwise, for each state $x \in \mathcal{X}$. In other words, $K \in \mathcal{S}$ is a fixed point of $A$ if and only if

$$\mathcal{S}(x, K(x)) = K(x) \forall x \in \mathcal{X}. \quad (5)$$

**General uniqueness conditions**

The previous insights will permit us to establish many additional properties of ME not derived in Morand & Reffett (2007). In particular, we will provide sufficient conditions under which the economy has a unique ME. To prepare these results, let $\Phi : \mathcal{X} \times \mathbb{R}^+ \rightarrow \mathbb{R}$,

$$\Phi(x, k^\prime) := k^\prime - \mathcal{S}(x, k^\prime). \quad (6)$$

By Lemma 2.1, $\Phi$ is measurable. Using the result (5), it follows that for each $x \in \mathcal{X}$, the value $k^* = K^*(x)$ of any fixed point $K^* \in \mathcal{S}$ of $A$ must be a zero of $\Phi(x, \cdot)$, i.e.,

$$\Phi(x, k^*) = 0. \quad (7)$$

Equation (7) is precisely the condition employed in de la Croix & Michel (2002) in a deterministic setting. It is also equivalent to – in fact, merely a re-statement of – the self-confirming expectations approach in Wang (1993). Define the correspondence

$$\Psi : \mathcal{X} \Rightarrow \mathbb{R}^+, \quad \Psi(x) := \{k^\prime \in \mathbb{R}^+ | \Phi(x, k^\prime) = 0\}. \quad (8)$$

Then, determining a ME is equivalent to finding a measurable selection of $\Psi$, i.e., a measurable function $\mathcal{K} : \mathcal{X} \rightarrow \mathbb{R}^+$ such that $\mathcal{K}(x) \in \Psi(x)$ for all $x \in \mathcal{X}$. Clearly, a necessary condition for ME to exist is that $\Psi$ be non-empty valued. It is also clear that if $\Psi$ is single-valued, i.e., a function, then there can be at most one ME. In this latter case, the next result shows that $\Psi$ will automatically be measurable, i.e., a unique ME exists. On the other hand, if for each $x \in \mathcal{X}$ the map $\Phi(x, \cdot)$ has at most one zero, there can be at most one ME. Thus, a sufficient condition for uniqueness is that $\Phi(x, \cdot)$ be strictly monotonic for all $x \in \mathcal{X}$. The following final result of this section summarizes these insights which will be key for the uniqueness result derived in the next section.

**Lemma 2.2**

Let $\Phi : \mathcal{X} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ defined in (6) be measurable. Then, the following holds:

(i) $\mathcal{E}$ has a unique ME, if and only if $\Phi(\hat{x}, \cdot)$ has a unique zero $\hat{k} > 0$ for each $\hat{x} \in \mathcal{X}$.

(ii) If $\Phi(\hat{x}, \cdot)$ has at most one zero for each $\hat{x} \in \mathcal{X}$, then $\mathcal{E}$ has at most one ME.

### 3 Uniqueness of Equilibrium

**Uniqueness of ME**

We are now in a position to establish our main results. The first one is the following theorem which lists sufficient conditions under which the economy has at most one ME.

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3 This very special structure is unique to overlapping generations models with two-period lived consumers. In most macroeconomic models with multi-period or infinitely-lived consumers, the operator $A$ will vary with the entire function $\mathcal{K}$, i.e., the value $(AK)(w)$ depends on the entire function $\mathcal{K}$. 

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Theorem 1
Let Assumptions 1 and 2 be satisfied. Then, each of the following restrictions is sufficient for the economy $\mathcal{E}$ to have at most one ME:

(i) $U(c^g, c^o) = u(c^g) + v(c^o)$ where $v$ satisfies $\frac{v''(c^o)}{v'(c)} \geq -1$.

(ii) $U_{12} \geq 0$ and $k \mapsto k\mathcal{R}(k, \varepsilon)$, $k > 0$ is weakly increasing for all $\varepsilon \in \mathcal{E}$.

(iii) $\frac{\varepsilon U_{22}(c^g, c^o)}{U_{12}(c^g, c^o)} > -1$ for all $(c^g, c^o) \geq 0$ and $k \mapsto \mathcal{R}(k, \varepsilon)$ is differentiable for all $\varepsilon \in \mathcal{E}$

where $\mathcal{R}_1 < 0$. In addition, either $U_{21} \leq 0$ or $\frac{\mathcal{R}_2(k, \varepsilon)^k}{\mathcal{R}(k, \varepsilon)} \geq -4$ for all $k > 0$, $\varepsilon \in \mathcal{E}$.

The hypotheses of Theorem 1 are satisfied for a broad class of economies. Condition (i) generalizes (i) to the non-additive case and holds, e.g., for Cobb-Douglas utility $E$ for the economy $\mathcal{E}$.

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1. $\mathcal{R}_1 \geq 0$ and $k \mapsto k\mathcal{R}(k, \varepsilon)$, $k > 0$ is weakly increasing for all $\varepsilon \in \mathcal{E}$.

2. $\frac{\varepsilon U_{22}(c^g, c^o)}{U_{12}(c^g, c^o)} > -1$ for all $(c^g, c^o) \geq 0$ and $k \mapsto \mathcal{R}(k, \varepsilon)$ is differentiable for all $\varepsilon \in \mathcal{E}$

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The hypotheses of Theorem 1 are satisfied for a broad class of economies. Condition (i) holds, e.g., if second-period utility displays constant relative risk aversion $0 < \theta \leq 1$, i.e.,

$v(c) = \frac{\gamma}{\gamma - 1} (c^{1-\theta} - 1)$, $\gamma > 0$ or under CES utility $v(c) = [1 - \theta + \theta c^\theta]^{1/\theta}$, $0 < \theta < 1$ if $0 \leq g < 1$. Also note that the restriction (i) is precisely Assumption 4 in Morand & Reffett (2007). This shows that the ME in their model is in fact unique. Thus, their findings can considerably be strengthened if the additional properties of the operator $A$ identified above are exploited.

Condition (ii) in Theorem 1 is the natural extension of the uniqueness condition in Wang (1993) to the present more general setting. For the case with a classical production function $f$, it holds, e.g., if $f$ is of the CES form $f(k, \varepsilon) = \varepsilon g(k)$ where $g(k) = [1 - \alpha + \alpha k^\theta]^\frac{1}{\theta}$, $0 < \alpha < 1$ and $0 \leq g < 1$ where $g = 0$ gives a Cobb-Douglas technology. The assumption of $U$ being supermodular ($U_{12} \geq 0$) is imposed throughout in Morand & Reffett (2007).

Finally, under the additional differentiability condition, (iii) permits to relax (ii) (which would imply $\frac{\mathcal{R}_2(k, \varepsilon)^k}{\mathcal{R}(k, \varepsilon)} \geq -1$) while imposing an additional restriction on $U$. The latter generalizes (i) to the non-additive case and holds, e.g., for Cobb-Douglas utility $U(c^g, c^o) = (c^g)^\alpha (c^o)^\beta$, $\alpha, \beta > 0$. Furthermore, for the classical CES production function mentioned above, $\frac{\mathcal{R}_2(k, \varepsilon)^k}{\mathcal{R}(k, \varepsilon)} = \frac{kg''(k)}{g'(k)} = \frac{-\alpha (1 - \theta) k^{-\alpha - \theta}}{1 - \alpha + \alpha k^\theta}$ which implies that the restriction on $\mathcal{R}$ in (iii) holds if $\theta \geq -3$. The latter could further be relaxed if an upper bound on $k$ such as $k < f(k, \varepsilon^{\text{max}})$ – which is done in Morand & Reffett (2007) – is imposed.

As a general insight, Theorem 1 shows that restrictions either on the consumption side (condition (i)) or the production side (condition (ii)) alone are already sufficient to induce a unique ME once it exists. It also reveals that the elasticities of the capital return function $\mathcal{R}$ and second-period marginal utility are key to the uniqueness of equilibrium while neither the wage function $\mathcal{W}$ nor the marginal utility of first-period consumption nor the transition $Q$ play a crucial role. As a consequence, multiple ME can occur only if capital income decreases very rapidly and the marginal utility of second-period consumption is very elastic. Also note that Theorem 1 does not ensure the existence of a ME. Imposing the additional restrictions of Morand & Reffett (2007), existence follows directly from their results.

Smoothness and monotonicity of ME
In cases where the ME is unique, one may ask which additional properties of $\mathcal{K}$ can be inferred. In Wang (1993), the factor pricing functions $\mathcal{R}$ and $\mathcal{W}$ are both $C^1$ which implies that the map $\mathcal{K}$ is also $C^1$ and strictly increasing in his model. In the present case, a similar result holds in the sense that $\mathcal{K}$ essentially inherits the properties of the factor pricing functions. The result needs the following additional

Assumption 3
For all $\varepsilon \in \mathcal{E}$, $k \mapsto \mathcal{R}(k, \varepsilon)$, $k > 0$ satisfies $\lim_{k \to 0} \mathcal{R}(k, \varepsilon) = \infty$ and $\lim_{k \to 0} k \mathcal{R}(k, \varepsilon) < \infty$. 

6
Theorem 2
Let Assumptions 1, 2, and 3 and any of the hypotheses (i), (ii), or (iii) of Theorem 1 be satisfied. In addition, suppose $k \mapsto W(k, \varepsilon)$ and $k \mapsto R(k, \varepsilon)$ are continuous for all $\varepsilon \in \mathcal{E}$. Then the economy $\mathcal{E}$ has a unique ME $K \in \mathcal{S}$ with the following properties:

(i) For all $\varepsilon \in \mathcal{E}$, $k \mapsto K(k, \varepsilon)$, $k > 0$, is continuous. It is increasing if $U_{21} \geq 0$.

(ii) If $W$ and $R$ are continuous and $Q$ has the Feller property, then $K$ is continuous.

(iii) If for all $\varepsilon \in \mathcal{E}$ $k \mapsto W(k, \varepsilon)$ and $k \mapsto R(k, \varepsilon)$ are $C^1$, then $k \mapsto K(k, \varepsilon)$ is $C^1$.

(iv) If, in addition to (iii), shocks are i.i.d., $E$ is an interval, and $W$ is $C^1$, then $K$ is $C^1$.

Further, if $W_2 > 0$, then $K_2 > 0$.

Uniqueness of SE
The previous discussion revolved around whether Markov equilibria are unique. This raises the question of whether there are other equilibria, i.e., recursive equilibria on a larger state space or non-recursive SE. A striking feature of the equilibrium structure from Section 2 is that the savings function $S$ is independent of how the forecast $k_{t+1}$ was obtained. In other words, it is independent of any hypothesized law of motion and just depends on the value predicted for the following period. As a consequence, any equilibrium process – recursive or not – satisfies $k_{t+1} \in \Psi(k_t, \varepsilon_t)$ for all $t \geq 0$. This shows that if the ME is unique, i.e., $\Phi(x, \cdot)$ has a unique zero for all $x \in X$, each state $x_t$ has a unique continuation value $k_{t+1}$ which is precisely the value determined by the ME. Thus, the capital process $\{k_{t+1}\}_{t \geq 0}$ and all other equilibrium variables are uniquely determined. As a consequence, the equilibrium is in fact unique whenever the ME is unique. These insights give the following final

Theorem 3
Suppose the economy $\mathcal{E}$ has a unique ME. Then, this is also the unique SE of $\mathcal{E}$.

4 Conclusions
Our analysis provides sufficient conditions under which a broad class of OLG economies with stochastic non-classical production and two-period lived consumers has at most one ME. The conditions obtained are quite general and should be easy to verify in applied work as they are stated directly on the primitives of the model. One aspect not discussed in this paper is whether the ME gives rise to a Stationary Markov Equilibrium (SME) corresponding to an invariant distribution on the state space. Using different techniques, this issue is studied at length in Wang (1994) and Morand & Reffett (2007). As the restrictions in this paper are weaker than those in Morand & Reffett (2007), their findings remain directly applicable in our setup if their additional restrictions are imposed.

A Mathematical Appendix
Proof of Lemma 2.4. Let $U := \{(w, s) \in \mathbb{R}^2_{++} | s < w\}$ and $v : \mathbb{R}^2_{++} \times U \rightarrow \mathbb{R}$, $v(r, w, s) := -U_1(w - s, sr) + rU_2(w - s, sr)$. Define $V := \mathbb{R}^2_{++} \times \mathcal{E} \times U$ and the map $V : V \rightarrow \mathbb{R}$

$$V(k', \varepsilon, w, s) := \int_{\mathcal{E}} v(R(k', \varepsilon'), w, s)Q(\varepsilon, d\varepsilon').$$

(A.1)
By the arguments from Section 2, \( V(k', \varepsilon, w, \cdot) : [0, 0, w] \mapsto \mathbb{R} \) has a unique zero \( 0 < s < w \) for each \((k', \varepsilon, w) \in S := \mathbb{R}_+ \times \mathcal{E} \times \mathbb{R}_+\), which may be written as a function \( \delta : S \mapsto \mathbb{R}_+ \), \( s = \delta(k', \varepsilon, w) \). It suffices to show that \( \delta \) is measurable. For \( n \geq 1 \), define \( S_n := [1/n, n] \times \mathcal{E} \times [1/n, n] \subset S \). Observe that \( S_n \) is an increasing \( (S_n \subset S_{n+1}) \) sequence of measurable \((S_n \in \mathcal{B}(S))\) subsets of \( S \) that converges to \( S = \bigcup_{n \geq 1} S_n \). Thus, by Lemma A.1, it suffices to show that each \( \delta_n := \delta|_{S_n} : S_n \mapsto [0, n] \) is measurable. Observe that both the domain and range of \( \delta_n \) being closed subsets of complete separable metric spaces are complete separable metric spaces. Using Buckley (1974, §3, Propositions 1 and 6), \( \delta_n \) is measurable if and only if \( \text{graph}(\delta_n) \) is a measurable subset of \( S_n \times [0, n] \), i.e., \( \text{graph}(\delta_n) \in \mathcal{B}(S_n \times [0, n]) \). For \( n \geq 1 \), define \( V_n := \{(k', \varepsilon, w, s) \in [1/n, n] \times \mathcal{E} \times [1/n, n] \mid s < w \} = V_n(S_n \times [0, n]) \) which is a measurable subset of \( V \). Consider the restriction \( V_n := V_n : V_n \mapsto \mathbb{R} \) which is measurable as the restriction of a measurable map \( V \) to a measurable subset of its domain. As \( \{0\} \) is a measurable subset of \( \mathbb{R} \) this implies that \( V_n^{-1}(\{0\}) = \{(k', \varepsilon, w, s) \in V_n \mid V_n(k', \varepsilon, w, s) = 0\} = \{(k', \varepsilon, w, s) \in S_n \times [0, n] \mid s = \delta_n(k', \varepsilon, w)\} = \text{graph}(\delta_n) \) is a measurable subset of \( V_n \), i.e., \( \text{graph}(\delta_n) \in \mathcal{B}(V_n) \). Since \( V_n \) is a measurable subset of \( S_n \times [0, n], \mathcal{B}(V_n) \subset \mathcal{B}(S_n \times [0, n]) \) which implies that \( \text{graph}(\delta_n) \in \mathcal{B}(S_n \times [0, n]) \). 

**Proof of Lemma 2.2.** (i) Let \( \mathcal{K} \) be the unique ME. Then, \( \mathcal{K} \) is a measurable selection of \( \Psi \) defined in (9). By contradiction, suppose for some \( \hat{x} \in X, \Phi(\hat{x}, \cdot) \) has at least two zeroes, say \( k_1 \) and \( k_2 \). W.l.o.g., suppose \( k_1 = \mathcal{K}(\hat{x}) \). Then, the function \( \mathcal{K}_2 : X \mapsto \mathbb{R}_+ \),

\[
\mathcal{K}_2(x) := \mathcal{K}(x) + 1_{\{\hat{x}\}}(x)(k_2 - k_1) = \begin{cases} k_2 & x = \hat{x} \\ \mathcal{K}(x) & \text{otherwise} \end{cases}
\]

is measurable and \( \mathcal{K}_2(x) \in \Psi(x) \) for all \( x \in X \), i.e., is another ME, contradicting uniqueness. Conversely, suppose \( \Phi(\hat{x}, \cdot) \) has a unique zero for all \( \hat{x} \in X \). In this case, the correspondence \( \Psi : X \mapsto \mathbb{R}_+ \) whose domain \( X = \mathbb{R}_+^2 \times \mathcal{E} \) is the product of an open and a closed subset of \( \mathbb{R} \) which is a Polish space. As open and closed subsets of Polish spaces are Polish and so are their product, both the domain and range of \( \Psi \) are Polish spaces. By Aliprantis & Border (2007, Theorem 12.28, p. 450), \( \Psi \) is measurable if and only if \( \text{graph}(\Psi) \) is a measurable subset of \( X \times \mathbb{R}_+ \), Measurability of \( \Phi : X \times \mathbb{R}_+ \mapsto \mathbb{R} \) implies that \( \Phi^{-1}(\{0\}) = \{(x, k') \in X \times \mathbb{R}_+ \mid \Phi(x, k') = 0\} = \{(x, k') \in X \times \mathbb{R}_+ \mid k' = \Psi(x)\} = \text{graph}(\Psi) \) is a measurable subset of \( X \times \mathbb{R}_+ \).

(ii) If for some \( \hat{x} \in X, \Phi(\hat{x}, \cdot) \) fails to have a zero, then \( \Psi(\hat{x}) = \emptyset \). Then any ME is a measurable selection of \( \Psi \), there will be no ME in this case. If, for all \( \hat{x} \in X, \Phi(\hat{x}, \cdot) \) has precisely one zero, then \( X \) exists by (i).

**Proof of Theorem 4.** Let \( x = (k, \varepsilon) \in X \) and \( w := \mathcal{W}(k, \varepsilon) > 0 \) be arbitrary but fixed. Using Lemma 2.2(ii), we show that the map \( \Phi(x, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R} \) defined in (3) has at most one zero \( k' > 0 \). Define \( \delta \) as in the proof of Lemma 2.1. Then, using (3) and (A.1)

\[
\Phi(x, k') = 0 \iff k' = \delta(k', \varepsilon, w) \iff V(k', w, \varepsilon, k') = 0. \tag{A.2}
\]

Thus, letting \( G(w, \varepsilon, k') := V(k', w, \varepsilon, k') \) with \( V \) defined as in (A.1), it suffices to show that \( G(w, \varepsilon, \cdot) : [0, w] \mapsto \mathbb{R} \) is strictly monotonic. Defining

\[
g(w, k', \varepsilon') := -U_1(w - k', k' \mathcal{R}(k', \varepsilon')) + \mathcal{R}(k', \varepsilon')U_2(w - k', k' \mathcal{R}(k', \varepsilon')) \tag{A.3}
\]

the function \( G \) may be written as

\[
G(w, \varepsilon, k') = \int_{\varepsilon} g(w, k', \varepsilon')Q(\varepsilon, d\varepsilon'). \tag{A.4}
\]
As integration preserves monotonicity, it suffices to show that $k' \rightarrow g(w, k', \varepsilon')$ is strictly monotonic – in fact, decreasing – in each of the three cases below.

(i) Under the hypotheses, the map $g$ in (A.3) takes the form  
\[ g(w, k', \varepsilon') = -w'(w - k') + f(k', \mathcal{R}(k', \varepsilon')) \]  
where $f: \mathbb{R}^2_+ \rightarrow \mathbb{R}_+$, $f(x, y) := yv'(yx)$. The first term in (A.5) is a strictly decreasing function of $k'$. Thus, it suffices to show that $k' \rightarrow f(k', \mathcal{R}(k', \varepsilon'))$ is decreasing for all $\varepsilon' \in \mathcal{E}$, which follows directly from $f_1(x, y) = y^2v''(yx) < 0 \leq f_2(x, y) = v'(yx) + yxv''(yx)$ and $k' \rightarrow \mathcal{R}(k', \varepsilon')$ being strictly decreasing by Assumption II.

(ii) Under the hypotheses, it is straightforward to show that $k' \rightarrow g(w, k', \varepsilon')$, $0 < k' < w$ defined in (A.3) is strictly decreasing for all $(w, \varepsilon') \in \mathbb{R}_+ \times \mathcal{E}$.

(iii) Under the additional hypothesis, the function $k' \rightarrow g(w, \varepsilon', k')$ defined in (A.3) is continuously differentiable for all $(w, \varepsilon') \in \mathbb{R}_+ \times \mathcal{E}$ and so is $G$. As differentiation and integration over a compact set may be interchanged, it suffices to show that $g_3 < 0$. Dropping the arguments for convenience, the derivative of (A.3) may be written as  
\[ g_3(w, \varepsilon', k') = \mathcal{R}_1 [U_2 + k'\mathcal{R}U_{22}] + U_{11} - 2\mathcal{R}U_{21} + \mathcal{R}^2U_{22} - U_{21}k'\mathcal{R}_1. \]  
The first term is strictly negative as $\mathcal{R}_1 < 0$ and the bracketed term is strictly positive by assumption. Suppose first that $U_{11} \leq 0$. Then, the last term in (A.3) is negative as well, so we need to show that $U_{11} - 2\mathcal{R}U_{21} + \mathcal{R}^2U_{22} < 0$ or, using $U_{ii} < 0$ for $i \in \{1, 2\}$  
\[ M := |U_{11}| - 2\mathcal{R}|U_{21}| + \mathcal{R}^2|U_{22}| \geq 0. \]

The concavity of $U$ implies a negative semi-definite Hessian matrix, so $U_{11}U_{22} \geq U_{12}^2$, which may be restated as $|U_{12}| \leq |U_{11}|\frac{1}{2}|U_{22}|\frac{1}{2}$. Substituting this result into (A.7) gives  
\[ M \geq |U_{11}| - 2\mathcal{R}|U_{11}|\frac{1}{2}|U_{22}|\frac{1}{2} + \mathcal{R}^2|U_{22}| = (|U_{11}|\frac{1}{2} - \mathcal{R}|U_{22}|\frac{1}{2})^2 \geq 0. \]

Second, suppose that $k'\mathcal{R}_1 \geq -4\mathcal{R}$. If $U_{21}(w - k', k'\mathcal{R}(k', \varepsilon')) \leq 0$, the argument of the previous step remains unchanged, so suppose $U_{21}(w - k', k'\mathcal{R}(k', \varepsilon')) > 0$. Then,  
\[ U_{11} - 2\mathcal{R}U_{21} + \mathcal{R}^2U_{22} - U_{21}k'\mathcal{R}_1 \leq U_{11} + 2\mathcal{R}U_{21} + \mathcal{R}^2U_{22} = -M \]

with $M$ defined in (A.7). As shown before, $M \geq 0$, which proves the claim.

**Proof of Theorem 2** Define $g$ and $G$ as in (A.3) and (A.4). Under the hypotheses, $g$ is a continuous function in $k'$. As integration over a compact set preserves continuity, $G$ is continuous in $k'$ for all $(w, \varepsilon) \in \mathbb{R}_+ \times \mathcal{E}$. The existence of a ME will follow from Lemma 2.2 (i) if we show that $G(w, \varepsilon, \cdot)$ has a unique zero for all $(w, \varepsilon) \in \mathbb{R}_+ \times \mathcal{E}$.

By Assumption 3, $\lim_{y \to x} g(w, \varepsilon', k) = -\infty$ and $\lim_{k \to \infty} g(w, \varepsilon', k) = \infty$ for all $\varepsilon' \in \mathcal{E}$ which implies $\lim_{y \to x} G(w, \varepsilon, k) = -\infty$ and $\lim_{k \to \infty} G(w, \varepsilon, k) = \infty$ for all $(w, \varepsilon) \in \mathbb{R}_+ \times \mathcal{E}$. Continuity of $G(w, \varepsilon, \cdot)$ thus ensures existence of a zero which is necessarily unique by monotonicity and defined by an implicit function $K: \mathbb{R}_+ \times \mathcal{E} \rightarrow \mathbb{R}_+$. By Lemma 2.2, the function $K : \mathbb{R} \rightarrow \mathbb{R}_+, K(k, \varepsilon) := \mathcal{K}(W(k, \varepsilon), \varepsilon)$ is the unique ME. Clearly, if $U_{12} \geq 0$, then $g$ and $G$ are both strictly increasing in $w$, which implies that $\tilde{K}(\cdot, \varepsilon)$ is strictly increasing. It then follows from Assumption 3 (ii) that $K(\cdot, \varepsilon)$ is increasing.
(i) Let $U := \{(w, k') \in \mathbb{R}^2_{+} | 0 < k' < w\}$. For $\varepsilon \in \mathcal{E}$, let $G^\varepsilon : U \rightarrow \mathbb{R}$, $G^\varepsilon(w, k) := G(w, \varepsilon, k)$. Under the hypotheses, $G^\varepsilon$ is a continuous function as the integrand is continuous and integration over a compact set preserves continuity. As shown in the proof of Theorem \ref{thm:continuity}, for each $w > 0$ the map $G^\varepsilon(w, \cdot) : [0, w] \rightarrow \mathbb{R}$ is strictly decreasing and, therefore, has a unique zero determined by some map $\hat{K}^\varepsilon : \mathbb{R}^+_+ \rightarrow \mathbb{R}^+_+$. We show that $\hat{K}^\varepsilon$ is continuous. For $n > 1$, choose a number $0 < \delta_n < \frac{1}{2n}$. Define the compact set $\mathcal{U}_n := \{(w, k) \in \mathbb{R}^2_{+} | 1/n \leq w \leq n, \delta_n \leq k \leq w - \delta_n\}$ and consider the restriction $G^\varepsilon_n := G^\varepsilon_{\mid \mathcal{U}_n} : \mathcal{U}_n \rightarrow \mathbb{R}$. Clearly, $G^\varepsilon_n$ is continuous as the restriction of a continuous function to a subset of its domain. We seek to determine $\delta_n$ such that each $G^\varepsilon_n(w, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ has a uniquely zero for all $w \in [1/n, n]$ determined by $\hat{K}^\varepsilon_n : [1/n, n] \rightarrow [0, n]$. Then, $\hat{K}^\varepsilon_n$ will be the restriction $\hat{K}^\varepsilon$ to $[1/n, n]$. Recall that $G^\varepsilon_n(w, \cdot)$ is continuous and strictly decreasing for all $w \in [1/n, n]$. Thus, $\hat{K}^\varepsilon_n$ is well-defined if $G^\varepsilon_n(w, \delta_n) < 0 < G^\varepsilon_n(w, w - \delta_n)$ for all $w \in [1/n, n]$. For $\delta_n > 0$, define $G^\varepsilon_{\max}(\delta_n) := \max_w \{G^\varepsilon(w, w - \delta_n) | w \in [1/n, n]\}$ and $G^\varepsilon_{\min}(\delta_n) := \min_w \{G^\varepsilon(w, \delta_n) | w \in [1/n, n]\}$ which are well-defined due to continuity of $G^\varepsilon$ and compactness of $[1/n, n]$. Note that $G^\varepsilon_{\max}$ is strictly increasing in $\delta_n$ and $\lim_{\delta_n \rightarrow 0} G^\varepsilon(w, w - \delta) = -\infty$ for all $w > 0$ implies $G^\varepsilon_{\max}(\delta_n) < 0$ for $\delta_n$ sufficiently small. Likewise, $G^\varepsilon_{\min}$ is strictly decreasing in $\delta_n$ and $\lim_{\delta_n \rightarrow 0} G^\varepsilon(w, \delta) = \infty$ for all $w > 0$, implies $G^\varepsilon_{\min}(\delta_n) > 0$ for $\delta_n$ sufficiently small. Thus, choosing $\delta_n$ small enough such that $G^\varepsilon_{\max}(\delta_n) < 0 < G^\varepsilon_{\min}(\delta_n)$ implies $G^\varepsilon(w, w - \delta_n) < 0 < G^\varepsilon(w, \delta_n)$ for all $w \in [1/n, n]$. Then, for each $w \in [1/n, n]$ there exists a unique zero $k_0 \in [\delta_n, w - \delta_n]$ of $G^\varepsilon_n(w, \cdot)$ determined by the function $\hat{K}^\varepsilon_n : [1/n, n] \rightarrow [0, n]$. We show that each $\hat{K}^\varepsilon_n$ is continuous. By Aliprantis & Border (2007, p.51, Theorem 2.58), it suffices to show that graph($\hat{K}^\varepsilon_n$) is a closed subset of $[1/n, n] \times [0, n]$. As $\{0\}$ is a closed subset of $\mathbb{R}$, continuity of $G^\varepsilon_n$ implies that ($G^\varepsilon_n)^{-1}(\{0\})$ is a closed subset of $\mathcal{U}_n$. But ($G^\varepsilon_n)^{-1}(\{0\}) = \{(w, k) \in \mathcal{U}_n | G^\varepsilon_n(w, k) = 0\} = \{(w, k) \in \mathcal{U}_n | k = K^\varepsilon(w)\} = \text{graph}(\hat{K}^\varepsilon_n)$. Thus, graph($\hat{K}^\varepsilon_n$) is a closed subset of $\mathcal{U}_n$. As $\mathcal{U}_n$ is a closed subset of $[1/n, n] \times [0, n]$, graph($\hat{K}^\varepsilon_n$) is also closed in $[1/n, n] \times [0, n]$. Now let $\hat{K}^\varepsilon_n$ be the restriction of $\hat{K}^\varepsilon_n$ to the open subset $S_n := [1/n, n[$. Clearly, each $\hat{K}^\varepsilon_n$ is continuous as the restriction of a continuous function to a subset of its domain. Moreover, $\hat{K}^\varepsilon_n$ is also the restriction of $\hat{K}^\varepsilon$ to $S_n$ and $\{S_n\}_{n \geq 1}$ is an increasing sequence of open subsets of $\mathbb{R}^+_+ = \bigcup_{n \geq 1} S_n$. Thus, continuity of $\hat{K}^\varepsilon$ for all $\varepsilon \in \mathcal{E}$ follows from Lemma A.1 and implies continuity of $\hat{K}(\cdot, \varepsilon)$.

(ii) Under the hypotheses, $g$ in (A.3) and $G$ in (A.4) are both continuous. One can now slightly modify the arguments of the previous step to show that $\hat{K}$ is a continuous function.

(iii) Let $\varepsilon \in \mathcal{E}$ be arbitrary but fixed. Under the additional hypotheses, the function $G^\varepsilon$ defined in (i) is $C^1$. Repeating the proof of Theorem \ref{thm:continuity} using differentiability, it is straightforward to show that $G^\varepsilon_2(w, k) > 0$ and an application of the implicit function theorem yields that $\hat{K}^\varepsilon$ defined in (i) is $C^1$ with derivative $\hat{K}^\varepsilon_1(w) = -G^\varepsilon_2(w, \hat{K}^\varepsilon(w))[G^\varepsilon_2(w, \hat{K}^\varepsilon(w))]^{-1}$.

(iv) If shocks are i.i.d., $G = G^\varepsilon$ and $\hat{K} = \hat{K}^\varepsilon$ are independent of $\varepsilon$. As shown in (iii), $\hat{K}$ is $C^1$ which implies that $\mathcal{K} = \hat{K} \circ \mathcal{W}$ is $C^1$. If, in addition, $W_2 > 0$ then clearly $\mathcal{K}_2 > 0$.

**Lemma A.1 (Auxiliary result)**

Let $f : \mathbb{X} \rightarrow \mathbb{Y}$ be a map between topological spaces $\mathbb{X}$ (with topology $\mathcal{T}_\mathbb{X}$) and $\mathbb{Y}$. Suppose there is an increasing sequence $\{\mathcal{X}_n\}_{n \geq 1}$ of open (measurable) subsets of $\mathbb{X}$ which converges to $\mathbb{X} = \bigcup_{n \geq 1} \mathcal{X}_n$. If each $f_n := f_{\mid \mathcal{X}_n} : \mathcal{X}_n \rightarrow \mathbb{Y}$ is continuous (measurable), then so is $f$.

**Proof:** Endow $\mathcal{X}_n$ with the relative topology $\mathcal{T}_n := \{U \cap \mathcal{X}_n | U \in \mathcal{T}_\mathbb{X}\}$. As $\mathcal{X}_n$ is open, $\mathcal{T}_n \subset \mathcal{T}_\mathbb{X}$ for all $n \geq 1$. Let $\mathcal{O}$ be an open subset of $\mathbb{Y}$. We have to show that $f^{-1}(\mathcal{O}) \in \mathcal{T}_\mathbb{X}$. 


As each $f_n$ is continuous, $f_n^{-1}(O) \in T_n \subset T_X$ and, therefore, $f_n^{-1}(O) \in T_X$ for all $n \geq 1$. We claim that $f^{-1}(O) = \bigcup_{n \geq 1} f_n^{-1}(O)$. Let $x \in f^{-1}(O)$ be arbitrary. Since $X = \bigcup_{n \geq 1} X_n$, there is some $X_m$ such that $x \in X_m$ and $f(x) = f_m(x) \in O$. Hence, $x \in f_m^{-1}(O) \subset \bigcup_{n \geq 1} f_n^{-1}(O)$ which shows that $f^{-1}(O) \subset \bigcup_{n \geq 1} f_n^{-1}(O)$. Conversely, let $x \in \bigcup_{n \geq 1} f_n^{-1}(O)$. Then, $x \in f_m^{-1}(O)$ for some $m$, i.e., $x \in X_m \subset X$ and $f(x) = f(x) \in O$. Conclude that $x \in f^{-1}(O)$ which implies $f^{-1}(O) \supset \bigcup_{n \geq 1} f_n^{-1}(O)$. Thus, $f^{-1}(O)$ is the union of open sets $f_n^{-1}(O) \in T_X$ which implies $f^{-1}(O) \in T_X$. The proof of measurability is analogous.

Acknowledgements

I would like to thank Martin Barbie, Tomoo Kikuchi, Caren Söhner, and John Stachurski for helpful discussions and comments.

References


