Bubbles and Crowding-in of Capital via a Savings Glut

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Abstract

This paper uncovers a mechanism by which bubbles crowd in capital investment. If capital formation is initially depressed by a binding credit constraint, a bubble triggers a savings glut. Higher returns in a new bubbly equilibrium attract additional savings, which are channeled to expand investment at the extensive margin leading to permanently higher capital, output, and wages. We demonstrate that crowding-in through this channel is a robust phenomenon that occurs along the entire time path.

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1 Introduction

On September 15 of 2008, the collapse of Lehman Brothers put a sudden end to a long-lasting stock market boom experienced by the US and many other industrialized countries. One year prior to this dramatic event, Ben Bernanke, Chairman of the Federal Reserve Bank reported a substantial increase in capital inflows to the US from high saving countries such as China, notably during 2003–2007 (cf. Bernanke 2007). Likewise, Alan Greenspan, Chairman of the Fed from 1987 to 2006 conjectured in 2010 that ‘a glut of excess intended saving’ might have caused or at least contributed to the boom in asset prices, in particular, housing prices prior to the financial crisis (cf. Greenspan (2010)). By making the US mortgage market awash with cash, Ferguson (2008) describes in his book how the Asian savings glut contributed to the subprime mortgage crisis in 2007.

These observations indicate that bubbly episodes may be caused or are at least accompanied by a massive increase in private savings invested in financial markets. The present paper offers a theoretical explanation of the relationship between a savings glut and asset bubbles and their impact on capital investment and the production side of the economy. Can a bubble generate additional resources by attracting new investors to the economy? Can these additional recourses stimulate productive investment in the economy?

Identifying the conditions under which asset bubbles emerge and understanding their consequences for the real economy is a subject of great importance in macroeconomics. The classical theory of asset bubbles initiated in a path-breaking paper by Tirole (1985) shows that bubbles occur naturally when there is capital overaccumulation and the growth rate exceeds the interest rate. Extensions of this model such as Kunieda (2008) or Michel & Wigniolle (2003) demonstrate that bubbles are also compatible with initial capital underaccumulation if a borrowing constraint induces a spread between the rates of returns on capital and the bubbly asset. In this case, the interest rate is no longer determined by the marginal product of capital, and only the rate of return on bubble has to be smaller than the growth rate for bubbles to exist.

A common feature of all Tirole-type models is that bubbles crowd out capital because they compete with productive investment in the portfolio of investors. But it is a well-documented empirical regularity that past episodes of bubbles witnessed investment booms when they emerged (c.f. Kindeleberger 1996). Recent major examples include the dot-com bubble in the US in the late 1990s, and real estate bubbles in Japan in the late 1980s and in the US in the first decade of the 21st century. In these instances, bubbles did
not crowd out but rather crowded in capital investment.

To explain this crowding-in puzzle, the literature has extended the classical approach of Tirole (1985) in various directions. Recent examples, among many others, are Caballero and Krishnamurthy (2006), Farhi and Tirole (2009), and Martin and Ventura (2012).\(^1\) Most of these models rely on mechanisms in which bubbles, by providing liquidity or collateral, help transfer resources towards productive investment. This liquidity or collateral effect of bubbles constitutes one important channel by which bubbles stimulate investment. A different approach is pursued in Caballero et al. (2006) who show that a temporary bubble coexists along a speculative growth path where the economy moves from a steady state with low capital to a steady state with high capital. In their model, however, a bubble unambiguously crowds out investment and leads to lower capital along the entire path relative to the bubbleless equilibrium.

The present paper proposes a different mechanism to explain why bubbles crowd in capital investment. The main driver in our model is a savings glut corresponding to a massive increase in private savings. This savings glut increases investment at the extensive margin, i.e., due to investors who newly join the market. Unlike in Caballero et al. (2006) a bubble is not a by-product but in fact the driver of the crowd ing-in phenomenon. We present a novel and intuitive economic mechanism which complements the existing approaches to explain the crowding-in puzzle.

The challenge is to build a model in which bubbles trigger a savings glut that fuels both bubbles and investment. We employ a version of Diamond’s OLG model with heterogeneous agents subject to a borrowing constraint.\(^2\) Crowding-in requires two conditions. First, a bubble must trigger a savings glut. Without the savings glut, the resources available for capital investment do not change and bubbles crowd out capital investment just as in Tirole (1985). Second, the borrowing constraint must be binding in the initial bubbleless steady state. Without the borrowing constraint, the return on loans is equal to the marginal product of capital. This implies that the return decreases with capital investment, which is not compatible with an increase in private savings.

In our model there are entrepreneurs, savers, and speculators. Speculators decide whether to join the market or not. If they do, they earn the entrepreneurial return with a certain probability. When a bubble appears, it competes with productive capital investment for

\(^1\)See Barlevy (2015) for a recent survey of the bubble literature.

\(^2\)The two country OLG model in Ikeda and Phan (2015) uses a similar restriction to study how capital flows from the South fuel bubbles in the North where labor productivity is higher.
the economy’s aggregate savings. This raises the interest rate earned by savers while lowering the entrepreneurial rate of return. The behavior of entrepreneurs is not much affected by this change as they only adjust their capital investment at the intensive margin. Neither is the behavior of savers affected who inelastically invest their entire income. Speculators, however, only provide resources to the economy if their expected return exceeds a certain threshold. Precisely this happens when a bubble causes a savings glut, adding new resources to the economy. Crowding-in occurs if this increase in savings overcompensates the resources absorbed by the bubble and expands capital investment.

The reduction in the spread in the returns earned by savers and entrepreneurs leads to adjustment of savings and investments at the extensive margin by speculators coming into a play when crowding-in occurs. The role of speculators in our framework is to trigger a savings glut by providing resources to the economy only if returns are sufficiently attractive. When they do not save, their wealth is not available—consumed or invested outside the economy. For this reason, we favor a broad interpretation of this group ranging from domestic investors with a high intertemporal elasticity of substitution in consumption to foreign investors who decide whether or not to invest in the domestic markets.

Our comparative statics results show that for crowding-in to occur both the probability for potential investors (speculators) to gain access to investment projects and the pledgeability of expected earnings must be sufficiently low. There is a tension between these two restrictions which requires a sufficiently tight borrowing constraint if the probability of having access to investment projects is high. This indicates that bubbles are more likely to crowd-in capital investment in a country with an immature financial market but abundant investment opportunities such as China or in other emerging economies with a more mature financial market but less abundant investment opportunities. When crowding-in occurs, the bubble may be able to overcome the borrowing constraint. For this to happen, either the borrowing constraint must not be too tight or the probability for speculators to gain access to investment projects must be sufficiently high.

The structure of the paper is as follows. Section 2 formulates the model and derives equilibrium conditions. Section 3 uncovers the recursive structure of equilibria and defines a state space. The existence of steady states and the scope for crowding-in are studied in Section 4. The economic mechanism of the model are inspected in Section 5. Section 6 studies the dynamics. Section 7 concludes. All proofs can be found in the appendix.
2 The model

2.1 Production sector

The production sector consists of a single firm which operates a constant-returns-to-scale technology to produce a consumption good using capital and labor as inputs. The production function in intensive form \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) is twice continuously differentiable and satisfies \( f'' < 0 = f(0) < f' \). At equilibrium labor supply will be constant and normalized to unity. Given the capital stock \( k_t > 0 \) and full depreciation of capital\(^3\), perfect competition in factor markets determines the wage and capital return in period \( t \) as

\[
\begin{align*}
  w_t &= f'(k_t) - k_t f''(k_t) \\
  \rho_t &= f''(k_t).
\end{align*}
\]  

(1)

2.2 Heterogenous agents

In each period \( t \geq 0 \), a continuum of young consumers is born whose mass is normalized to unity. Each of these consumers lives for two periods and supplies one unit of labor in the first period to earn labor income \( w_t > 0 \). Each generation consists of: (i) savers, (ii) entrepreneurs, and (iii) speculators, who differ in terms of their access to investment projects and consumption behavior.

Savers only consume in the second period of life and, therefore, wish to transfer their current wealth \( w_t \) into the next period. For this purpose, they supply loans to the credit market and purchase bubbly assets whose value at time \( t \) is \( b_t \geq 0 \). Both investments yield an identical return \( R_{t+1} > 0 \).

Entrepreneurs have access to investment projects, which transform final goods unit by unit into capital available in the next period. Entrepreneurs also consume only when old and, therefore, invest their entire wealth when young. In addition, they take loans in the credit market to finance their capital investment. The gross return earned by entrepreneurs on their income \( w_t \) is \( R_{t+1}^E \geq R_{t+1} \).

Speculators have to decide whether to consume or save their wealth \( w_t \). When they save, they gain access to investment projects with probability \( p \in (0,1) \) and otherwise behave

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\(^3\)As in the standard overlapping generations model, the capital dynamics are essentially the same even when we assume partial depreciation.
as a saver. The decision is irreversible and made ex-ante, i.e., before the uncertainty is revealed.\textsuperscript{4} Thus, it must be based on the ex-ante return

\[ R_{t+1}^S := pR_{t+1}^E + (1 - p)R_{t+1}. \]  

Speculators save their wealth if and only if \( R_{t+1}^S \) exceeds a threshold \( \rho > 0 \). There are various interpretations for this assumption. First, \( \rho \) may represent an outside investment opportunity (e.g., a foreign asset promising the return \( \rho \)). A second, alternative interpretation is that speculators are endowed with a linear utility function \( u(c^y, c^0) = c^y + \frac{1}{\rho}c^0 \) which would result in the same investment behavior. In what follows we simply treat \( \rho \) as a parameter of the model.

The role of \( p \) in (2) is to generate ex-ante uncertainty in the decision behavior of speculators. Ex-post, this parameter determines the share \( p \) of speculators who become investors while the remaining \( 1 - p \) become savers. As we will see below, this structure allows capital investment to expand at the extensive margin due to an increase in the population share of investors. This type of adjustment will play a crucial role in our mechanism to explain crowding-in.

### 2.3 Savings glut

The saving behavior of speculators is the driving force of the model. Thus, without loss of generality, the population shares of entrepreneurs and savers are assumed to be the same and equal to \( a \in (0, 1/2) \). Defining \( s^L := 2a \) and \( s^H = 1 \), the saving decision by speculators determines the aggregate saving rate

\[ s_t = \begin{cases} 
\frac{1}{2} & \text{if } R_{t+1}^S < \rho \\
1 & \text{otherwise}
\end{cases} \]  

Equation (3) is precisely what we call a **savings glut** in our economy. It represents an adjustment of aggregate savings at the extensive margin, i.e., due to more people investing if the return exceeds a certain threshold.

At the aggregate level, the total resources \( s_t w_t \) invested in period \( t \) are used to form the capital stock \( k_{t+1} \) of the following period and finance the current bubble \( b_t \). Thus,

\[ k_{t+1} = s_t w_t - b_t. \]  

\textsuperscript{4}Caballero and Krishnamurthy (2006) use a similar setup where investment opportunities are revealed after saving decisions are made.
One observes from (4) that the savings boost (3) has the potential to explain crowding-in of capital: If the injection of a bubble \( b_t \) causes the savings rate \( s_t \) to increase from \( s^L \) to \( s^H \), this may overcompensate the resources absorbed by the bubble and capital may increase relative to the bubbleless situation.

### 2.4 Financial friction

Let \( \alpha_t \) denote the share of consumers who run investment projects at time \( t \). These consumers will be called investors. The behavior of speculators determines the share of investors as

\[
\alpha_t = \begin{cases} 
  a & R^S_{t+1} < \rho \\
  a + p(1-2a) & \text{otherwise.}
\end{cases} \tag{5}
\]

Investors in period \( t \) take the wage \( w_t \) and returns on capital \( f'(k_{t+1}) \) and loans \( R_{t+1} \) as given and choose investment \( i \) to maximize expected profit. The objective function reads

\[
\Pi_{t+1}(i) = f'(k_{t+1})i - R_{t+1}(i - w_t). \tag{6}
\]

Similar to Matsuyama (2004), we assume that investors can credibly pledge only a fraction \( \lambda \in (0, 1) \) of expected earnings \( f'(k_{t+1})i \) to meet their repayment obligation \( R_{t+1}(i - w_t) \). Thus, the choice of \( i \) is made subject to the borrowing constraint

\[
R_{t+1}(i - w_t) \leq \lambda f'(k_{t+1})i. \tag{7}
\]

The parameter \( \lambda \) can be interpreted as a measure of financial market imperfections in the economy, with higher value corresponding to a lower degree of imperfection.\(^5\) The credit market friction constitutes the second key ingredient to our model.

### 2.5 Equilibrium

By (6), positive investment at equilibrium requires returns to satisfy the profitability constraint

\[
R_{t+1} \leq f'(k_{t+1}). \tag{8}
\]

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\(^5\)The simplest story to justify the assumption is that borrowers strategically default whenever the repayment obligation exceeds the default cost, which is proportional to the project revenue.
By (3) and (5), the share of consumers who behave as a saver can be written as \( s_t - \alpha_t \). Thus, the supply of loans is \((s_t - \alpha_t)w_t - b_t\) and the demand for loans is \(\alpha_t(i_t - w_t)\). This and (4) determine equilibrium investment

\[
i_t = \frac{s_t w_t - b_t}{\alpha_t} = \frac{k_{t+1}}{\alpha_t}.
\]  

(9)

Using (9) in (7) determines the equilibrium borrowing constraint

\[
R_{t+1}(k_{t+1} - \alpha_t w_t) \leq \lambda f'(k_{t+1}) k_{t+1}.
\]  

(10)

Perfect competition in the credit and capital markets implies that the return \( R_{t+1} \) adjusts until either the constraint (8) or (10) binds in equilibrium. Thus, given \( k_{t+1} \) determined by (4) we obtain

\[
R_{t+1} = \frac{f'(k_{t+1})}{k_{t+1} - \alpha_t w_t} \min \left\{ k_{t+1} - \alpha_t w_t, \lambda k_{t+1} \right\}.
\]  

(11)

Note that \( R_{t+1} < f'(k_{t+1}) \) if the borrowing constraint (10) is binding. Also observe that investors may invest part of their wealth in bubbles if the profitability constraint (8) is binding. This happens precisely if \( i_t < w_t \), i.e., \( k_{t+1} < \alpha_t w_t \). Using (9) in (6) the return earned by each entrepreneur is

\[
R_{t+1}^E = \frac{\Pi_{t+1}(i_t)}{w_{t+1}} = \frac{f'(k_{t+1})}{\alpha_t w_t} \max \left\{ \alpha_t w_t, (1 - \lambda) k_{t+1} \right\}.
\]  

(12)

Note that if the borrowing constraint (10) is binding, \( R_{t+1}^E > f'(k_{t+1}) \) and speculators strictly prefer running projects to supplying credit or purchasing bubbles.

Young savers lend in the credit market at the return \( R_{t+1} \) or purchase bubbles, which have a fixed supply normalized to one, at price \( b_t \) and sell them at price \( b_{t+1} \) in the next period. Given the return determined by (11), no-arbitrage implies that the bubble evolves as

\[
b_{t+1} = R_{t+1} b_t.
\]  

(13)

Unlike other models in the literature (e.g., Caballero & Krishnamurthy (2006)), the bubble in our economy never bursts.

The economy can be summarized as \((a, p, \lambda, \rho, f)\) plus initial conditions \( k_0 > 0 \) and \( b_0 \geq 0 \). Following is a general definition of equilibrium.

**Definition 2.1.** Given \( k_0 > 0 \) and \( b_0 \geq 0 \), an equilibrium is a sequence of non-negative values \( \{w_t, b_t, s_t, \alpha_t, k_{t+1}, R_{t+1}, R_{t+1}^E, R_{t+1}^S\}_{t \geq 0} \) which satisfies (1)-(5) and (11)-(13) for all \( t \geq 0 \).
3 Recursive equilibrium structure

In this section, we uncover the forward-recursive structure of equilibria and formulate an appropriate state space.

3.1 Determining the equilibrium mappings

Consider an arbitrary period \( t \geq 0 \) and let the current bubble \( b_t \geq 0 \) be determined by (13) and the capital stock \( k_t > 0 \) determining the wage \( w_t > 0 \) by (1) be given. In the following analysis, we will choose \( x_t := (w_t, b_t) \) as our state variable which takes values in some state space \( X \) to be specified below. While this is equivalent to the usual choice \( (k_t, b_t) \), it will considerably simplify the results. In particular, several equilibrium constraints and the stable manifold defined in Section 6.1 will become linear.

Given a candidate savings rate \( s_t \in \{s^L, s^H\} \), the variables determined by (4), (5), (11), (12) and (2) can be written as the following functions of the state variable \( x_t \):

\[
\begin{align*}
    k_{t+1} &= \mathcal{K}(x_t; s_t) := s_t w_t - b_t \quad (14a) \\
    \alpha_t &= a(s_t) := a + p(s_t - 2a) \quad (14b) \\
    R_{t+1} &= \mathcal{R}(x_t; s_t) := \frac{f'(k_{t+1})}{k_{t+1} - \alpha_t w_t} \min \left\{ k_{t+1} - \alpha_t w_t, \lambda k_{t+1} \right\} \quad (14c) \\
    R_{t+1}^E &= \mathcal{R}^E(x_t; s_t) := \frac{f'(k_{t+1})}{\alpha_t w_t} \max \left\{ \alpha_t w_t, (1 - \lambda)k_{t+1} \right\} \quad (14d) \\
    R_{t+1}^S &= \mathcal{R}^S(x_t; s_t) := p \mathcal{R}^E(x_t; s_t) + (1 - p) \mathcal{R}(x_t; s_t) \quad (14e).
\end{align*}
\]

By (14a), positivity of capital requires \( b_t < s_t w_t \), which imposes a first restriction on \( X \). A second restriction is that the savings rate \( s_t \) must be consistent with the savings behavior of speculators. For \( s_t = s^H \), this requires \( \mathcal{R}^S(w_t, b_t; s^H) \geq \rho \) while \( \mathcal{R}^S(w_t, b_t; s^L) < \rho \) must hold for \( s_t = s^L \) to be consistent. Suppose that precisely one of these conditions holds. Then, the consistent savings rate is determined by the mapping

\[
S : X \rightarrow \{s^L, s^H\}, \quad s_t = S(x_t) := \begin{cases} 
    s^H & \text{if } \mathcal{R}^S(x_t; s^H) \geq \rho \\
    s^L & \text{if } \mathcal{R}^S(x_t; s^L) < \rho.
\end{cases} \quad (15)
\]
Below we impose restrictions on $X$ such that at least one of the two qualifications holds. Using (15) we can write all variables determined by equations (14b–d) as functions of $x_t$ alone. Furthermore, using (14b–e) and (15), the new state $x_{t+1} = (w_{t+1}, b_{t+1})$ is determined by

$$\begin{align*}
w_{t+1} &= \Phi_1(w_t, b_t) := \mathcal{W} \circ K(w_t, b_t; S(w_t, b_t)) \\
b_{t+1} &= \Phi_2(w_t, b_t) := b_t \mathcal{R}(w_t, b_t; S(w_t, b_t))
\end{align*}$$

(16a, 16b)

where $\mathcal{W}(k) := f(k) - kf'(k)$. An initial state $x_0 \in X$ defines an equilibrium in the sense of Definition 2.1 if (16a,b) generates a sequence $\{x_t\}_{t \geq 0}$ of states which satisfy $x_t \in X$ for all $t \geq 0$. All other equilibrium variables then follow from (14b–e) and (15).

### 3.2 Defining a state space

The set $X$ of states $x_t$ for which a continuation value $x_{t+1}$ is well-defined decomposes into the set $X^H$ of states consistent with high savings $s_t = s^H$ and the set $X^L$ consistent with low savings $s_t = s^L$. Formally, recalling the additional restriction $b_t < s_t w_t$

$$\begin{align*}
X^H &:= \{ (w, b) \in \mathbb{R}^2_+ \mid b < s^H w, \mathcal{R}^S(w, b; s^H) \geq \rho \} \\
X^L &:= \{ (w, b) \in \mathbb{R}^2_+ \mid b < s^L w, \mathcal{R}^S(w, b; s^L) < \rho \}
\end{align*}$$

(17a, 17b)

Defining the state space $X := X^L \cup X^H$ ensures that each $x_t \in X$ has a continuation value $x_{t+1}$ which is unique whenever $x_t \in X \setminus (X^L \cap X^H)$. As mentioned in footnote 4, if $X^L \cap X^H \neq \emptyset$, the savings function (15) can be modified appropriately to induce a unique continuation value. Also observe that for (16a,b) to define a dynamical system, the mappings $\Phi = (\Phi_1, \Phi_2)$ must further be restricted to a subset $\overline{X} \subset X$ which is self-supporting, i.e., $\Phi(\overline{X}) \subset \overline{X}$. This is a feature typical of models with bubbles also present—though not explicitly discussed —in Tirole (1985). We will handle this issue in the following sections.

### 3.3 Regimes in the state space

The sets defined in (17a,b) can be partitioned into states where the borrowing constraint is binding and where it is not. By (14c), the borrowing constraint is binding in period $t$, if

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6For states where both $s^H$ and $s^L$ are consistent, the definition (15) would need to be modified to select a savings rate, for example, by a random selection.
and only if \( k_{t+1} - \alpha_t w_t > \lambda k_{t+1} \). Using (14a), this is equivalent to \( b_t < \gamma(s_t) w_t \) where

\[
\gamma(s) := s - \frac{\alpha(s)}{1 - \lambda}, \quad s \in \{s^L, s^H\}.
\]

(18)

We see that whether the borrowing constraint binds in a given savings regime depends entirely on the size of the ratio \( \frac{b_t}{w_t} \) relative to a constant \( \gamma(s) \). As a consequence, the borrowing constraint is more likely to bind if the bubble is small and vanishes if \( b_t \) is large, relative to the wage \( w_t \). If \( b_t \) is small, savings net of bubbles and thus investment are higher. This implies a higher loans-to-investment ratio making the borrowing constraint more likely to bind. In contrast, low savings net of bubbles decrease the loans-to-investment ratio which relaxes the borrowing constraint. Also observe that \( \gamma(s) < s \) and, possibly, \( \gamma(s) < 0 \). In the latter case, the borrowing constraint will never bind.

Thus, we obtain four regions depending on whether savings are high or low and the borrowing constraint is binding or not. Formally, \( X^H = X^H_B \cup X^H_N \) and \( X^L = X^L_B \cup X^L_N \) where

\[
X^H_B = \left\{ (w, b) \in X^H \mid b < \gamma(s^H) w \right\}, \quad X^L_B = \left\{ (w, b) \in X^L \mid b < \gamma(s^L) w \right\}
\]

\[
X^H_N = \left\{ (w, b) \in X^H \mid b \geq \gamma(s^H) w \right\}, \quad X^L_N = \left\{ (w, b) \in X^L \mid b \geq \gamma(s^L) w \right\}.
\]

(19)

We also let \( X_B := X^H_B \cup X^L_B \) and \( X_N := X^H_N \cup X^L_N \) the set of states where the borrowing constraint is binding and where it is not.

4 Steady state analysis

In this section, we analyze steady states of \( \Phi \), i.e., states \( \bar{x} \in X \) for which \( \Phi(\bar{x}) = \bar{x} \). We call a steady state \( \bar{x} = (\bar{w}, \bar{b}) \) bubbly if \( \bar{b} > 0 \) and bubbleless otherwise. In what follows, variables associated with a steady state \( \bar{x} \) are denoted by a bar superscript. An additional superscript zero identifies variables associated with a bubbleless steady state \( \bar{x}^0 = (\bar{w}^0, 0) \).

4.1 Existence of steady states

To obtain explicit results and closed-form solutions, the remainder of this paper assumes a Cobb-Douglas production technology \( f(k) := k^\theta, \ 0 < \theta < 1 \). The economy is then summarized by the list \( (a, p, \lambda, \rho, \theta) \). Under this additional restriction, we can derive explicit conditions under which steady states of either type (bubbleless or bubbly) exist in
the four regimes of the state space. The results are stated formally in Lemma A.1 and A.2 in Appendix A.1. The main findings are the following.

First, compared to similar models in the literature, additional restrictions are necessary for steady states of either type to exist. Generally speaking, the existence of a steady state in one of the four regimes defined in (19) requires consistency with (i) the borrowing constraint which may be binding or not and (ii) the savings rate which may be high or low. As in similar models such as Tirole (1985) or Kunieda (2008), the conditions under which a bubbly steady state exists can be stated in terms of the returns at the bubbleless steady state in the same regime. (If a bubbleless steady state in the same regime fails to exist, one should interpret the condition as a shadow return.) In particular, a bubble return less than unity is necessary to ensure existence of a bubbly steady state. Section 5 offers a detailed discussion of these conditions.

Second, both types of steady states may not be unique. However, if there are two bubbleless steady states, one will be in $X_B^L$ and the other one in $X_N^L$ or one will be in $X_B^H$ and the other one in $X_N^H$. Thus, in each of the sets $X_N$, $X_B$, $X_H$ and $X_L$ the map $\Phi$ possesses at most one bubbleless steady state. The same holds for bubbly steady states. Again, the potential multiplicity of steady states of each type—which is not observed in comparable OLG models with bubbles—is due to the interaction between the financial friction and the savings glut. It vanishes as soon as one of these ingredients is turned off (cf. Section 5).

### 4.2 Crowding-in

Crowding-in occurs if capital or, equivalently, the wage in a bubbly steady state is higher than in the bubbleless steady state. Formally, we have

**Definition 4.1.** Given a bubbleless steady state $\bar{x}^0 = (\bar{w}^0, 0)$ and a bubbly steady state $\bar{x} = (\bar{w}, \bar{b})$, $\bar{b} > 0$, we say that crowding-in occurs iff $\bar{w} > \bar{w}^0$.

The following result shows that both the savings glut and the borrowing constraint are essential for crowding-in. The economic intuition why this holds will be developed in the next section.

**Lemma 4.1.** The following conditions are necessary for crowding-in:

(i) $\bar{x}^0 \in X^L$ and $\bar{x} \in X^H$  

(ii) $\bar{x}^0 \in X_B$
Combining both insights from Lemma 4.1, a necessary condition for crowding-in is that \( \bar{x}^0 \in X_B^L \), i.e., the bubbleless steady state must lie in the low-savings regime and the borrowing constraint must be binding at \( \bar{x}^0 \). Further, \( \bar{x} \) must lie in the high-savings regime \( X_H \) such that the savings boost is necessary for crowding-in to occur. It follows that we can distinguish the two cases where the borrowing constraint is non-binding and binding at \( \bar{x}^0 \).

To obtain sufficient conditions for crowding-in between \( \bar{x}^0 \) and \( \bar{x} \), let \( \bar{k}^0 \) and \( \bar{k} \) be the associated steady state capital stocks. Then, crowding-in is equivalent to the condition \( f'(\bar{k}^0) > f'(\bar{k}) \).

As shown in Appendix A.1, the mapping \( \varpi^0 \) determines the capital return at the bubbleless steady state depending on the prevailing savings rate. Likewise, \( \varpi_B \) determines the capital return at the bubbly steady state if the borrowing constraint is binding (if the constraint is non-binding, the capital return at the bubbly steady state equals unity).

With these insights, we are now in a position to state our first main result which provides a complete characterization for crowding-in to occur in our model.

**Theorem 4.1.** Let \( \bar{x}^0 = (\bar{w}^0, 0) \) and \( \bar{x} = (\bar{w}, \bar{b}) \), \( \bar{b} > 0 \) be steady states of \( \Phi \). Then, crowding-in occurs if and only if one of the following conditions hold:

(i) \( \bar{x}^0 \in X_B^L, \bar{x} \in X_N^H, \) and \( \varpi^0(s^L) > 1 \)

(ii) \( \bar{x}^0 \in X_B^L, \bar{x} \in X_B^H, \) and \( \varpi^0(s^L) > \varpi_B(s^H) \).

As Theorem 4.1 is not stated in terms of the primitive parameters of the model, it raises the questions whether the set of economies which satisfy either conditions (i) or (ii) of Theorem 4.1 are non-empty and how they can be characterized. Let \( \mathcal{E} := \{(a, p, \lambda, \rho, \theta) \in \mathbb{R}^5 | 0 < a < \frac{1}{2}, 0 < p < 1, 0 < \lambda < 1, \rho > 0, 0 < \theta < 1\} \) denote the entire class of economies studied in this paper and \( \mathcal{E}_1 \subset \mathcal{E} \) and \( \mathcal{E}_2 \subset \mathcal{E} \) be the subclasses of economies which satisfy conditions (i) respectively (ii) of Theorem 4.1. Then, \( \mathcal{E}_1 \cup \mathcal{E}_2 \) is the class of economies for which crowding-in occurs. Note that \( \mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset. \)

\(^{7}\)This follows by uniqueness of the bubbly steady state in \( X^H = X_N^H \cup X_B^H \), cf. Lemma A.2. Alternatively, the conditions in Theorem A.1 imply \( \theta' \frac{1-\lambda}{\pi(a;p)} < 1 \) in \( \mathcal{E}_1 \) and \( \theta' \frac{1-\lambda}{\pi(a;p)} > 1 \) in \( \mathcal{E}_2 \).
The following result shows that the phenomenon of crowding-in is robust in the sense that it occurs on an open subclass of economies satisfying the assumptions in this paper. A detailed discussion of the parameter conditions defining the classes $\mathcal{E}_1$ and $\mathcal{E}_2$ is provided in Section A.2 in the appendix.

**Theorem 4.2.** Both classes $\mathcal{E}_1$ and $\mathcal{E}_2$ are non-empty and contain open subclasses of economies for which crowding-in occurs.

![Figure 1: Projections of the classes $\mathcal{E}_1$ and $\mathcal{E}_2$, $a = 0.21$, $\rho = 1$, $\theta = 0.33$.](image)

Figure 1 depicts a projection of the classes $\mathcal{E}_1$ and $\mathcal{E}_2$ into the $p - \lambda$ space. Additional details how the figure is constructed may be found in Section A.2.

### 4.3 Examples for crowding-in

The following examples illustrate the two cases (i) and (ii) of Theorem 4.1.

**Example 1:** $\bar{x}^0 \in X^L_B$, $\bar{x} \in X^H_N$.

The economy $(a, p, \lambda, \rho, \theta) = (0.21, 0.30, 0.25, 1.00, 0.33)$ satisfies the conditions in Theorem A.1 (i) and, therefore, belongs to $\mathcal{E}_1$. Thus, the economy has a bubbleless steady state $\bar{x}^0 = (\bar{w}^0, 0)$ in $X^L_B$ and a bubbly steady state $\bar{x} = (\bar{w}, \bar{b})$ in $X^H_N$. The values at the respective
steady states compute

\[
\begin{align*}
\bar{\omega}^0 &= 0.36 & \bar{k}^0 &= 0.15 & f'(\bar{k}^0) &= 1.17 & \bar{r}^0 &= 0.72 \\
\bar{\omega} &= 0.39 & \bar{k} &= 0.19 & f'(\bar{k}) &= 1 & \bar{i} &= 0.50 \\
\bar{R}^0 &= 0.59 & \bar{R}^{S,0} &= 0.94 & \bar{R}^{E,0} &= 1.76 & \bar{l} &= 0.07 \\
\bar{R} &= 1.00 & \bar{R}^S &= 1.00 & \bar{R}^E &= 1.00 & \bar{l} &= 0.04 \\
\end{align*}
\]

Here, \( \bar{r}^0 = \frac{\bar{\omega}^0}{\alpha(s^L)} \) and \( \bar{i} = \frac{\bar{k}}{\alpha(s^H)} \) is the investment and \( \bar{r}^0 = \alpha(s^L)(\bar{r}^0 - \bar{\omega}^0) \) and \( \bar{l} = \alpha(s^H)(\bar{l} - \bar{\omega}) \) the credit volume at the respective steady states which we compute for later reference.

While in principle the economy could have an additional co-existing bubbleless steady state \( \bar{x}^0 \in X^H_N \) and/or a bubbly steady state \( \bar{x} \in X^L_B \), we readily check that neither the conditions from Lemma A.1(i) nor from Lemma A.2(iv) are satisfied. Thus, the values computed are in fact the only steady states of the economy. The bubble in this example crowds in capital by 27\%.

**Example 2:** \( x^0 \in X^L_B, x \in X^H_B \).

The economy \( (a, p, \lambda, \rho, \theta) = (0.21, 0.30, 0.20, 1.00, 0.33) \) satisfies the conditions in Theorem A.1(ii) and, therefore, belongs to \( \mathcal{C}_2 \). Thus, the economy has a bubbleless steady state \( \bar{x}^0 = (\bar{\omega}^0, 0) \) in \( X^L_B \) and a bubbly steady state \( \bar{x} = (\bar{\omega}, \bar{b}) \) in \( X^H_B \). The steady state values compute

\[
\begin{align*}
\bar{\omega}^0 &= 0.36 & \bar{k}^0 &= 0.15 & f'(\bar{k}^0) &= 1.17 & \bar{r}^0 &= 0.72 \\
\bar{\omega} &= 0.38 & \bar{k} &= 0.19 & f'(\bar{k}) &= 1.02 & \bar{i} &= 0.48 \\
\bar{R}^0 &= 0.47 & \bar{R}^{S,0} &= 0.89 & \bar{R}^{E,0} &= 1.88 & \bar{l} &= 0.07 \\
\bar{R} &= 1.00 & \bar{R}^S &= 1.01 & \bar{R}^E &= 1.03 & \bar{l} &= 0.04 \\
\end{align*}
\]

Similar to the previous case, neither the conditions from Lemma A.1(i) nor from Lemma A.2(ii) hold. Thus, the steady states computed are the only ones of the economy. The bubble in this example crowds in capital by 20\%.

In both of the previous examples the return earned by entrepreneurs decreases while the one on loans rises. The final result of this section states that this result holds in general.

**Proposition 4.1.** Under crowding-in, the associated returns satisfy \( \bar{R}^E < \bar{R}^{E,0} \) and \( \bar{R} > \bar{R}^0 \).

Proposition 4.1 shows that crowding-in is necessarily accompanied by a reduction in the steady state return spread, i.e., \( \bar{R}^E - \bar{R} < \bar{R}^{E,0} - \bar{R}^0 \). The economic mechanism behind this will be clarified in Section 5.
4.4 Consumption and welfare

To study the long-run welfare effects of crowding-in we compare the consumption levels of savers and entrepreneurs at the respective steady states. An immediate consequence of Proposition 4.1 is that crowding-in makes savers better off since both the return on loans and the wage increase. Thus, their (second period) consumption satisfies $R\bar{w} > R^0\bar{w}^0$. The effect on entrepreneurial consumption, however, is ambiguous: This group will be better off only if the rise in the wage overcompensates the decline in the return on their investment such that $R^E\bar{w}^0 < R^E\bar{w}$. In both of the previous numerical examples, this is not the case. Thus, we infer that crowding-in does, in general, not lead to a Pareto-improvement in our model which would make all consumers better off. Also note that crowding-in shifts the distribution of aggregate consumption towards a more equal distribution by increasing the consumption share of savers and lowering the share received by entrepreneurs. In fact, the consumption distribution becomes uniform in case (i) of Theorem 4.1 while entrepreneurs continue to consume a higher share in case (ii).

An alternative welfare measure is aggregate consumption given by $f(k_t) - k_{t+1}$ in each period $t$. The ‘optimal’ capital stock $k^* > 0$ which maximizes the aggregate steady state consumption $C(k) := f(k) - k$ is therefore uniquely characterized by the familiar golden rule condition $f'(k^*) = 1$. The existence conditions from Theorem 4.1 and strict concavity of $C$ then imply directly that $f'(\bar{k}^0) > f'(\bar{k}) = f'(k^*)$ and $C(\bar{k}^0) < C(\bar{k}) = C(k^*)$ in case (i) while $f'(\bar{k}^0) > f'(\bar{k}) > f'(k^*)$ and $C(\bar{k}^0) < C(\bar{k}) < C(k^*)$ in case (ii). In either case, crowding-in raises the aggregate consumption which increases up to its optimal level in case (i) but remains below it in case (ii).

All of the previous effects reverse if the economy initially is in the bubbly steady state which (unexpectedly) collapses leading to a bubbleless state with a lower aggregate consumption and a consumption distribution more in favor of entrepreneurs in the long-run.

5 The mechanism

This section inspects the economic mechanisms of our model that generate crowding-in. Recall that our model has two main ingredients: a financial friction and a savings glut. We

8For this and the following arguments to hold, we take the interpretation that speculators are endowed with a linear utility function as specified in page 6.
will first study each role separately by turning off the other mechanism and then show how their interaction leads to crowding-in. For ease of notation, define the functions

\[ R_B^0(s) := \frac{\theta}{1-y} \frac{\lambda}{s-\alpha(s)} \]

\[ R_{E,0}^B(s) := \frac{\theta}{1-y} \frac{1-\lambda}{s-\alpha(s)} \]

\[ R_{S,0}^B(s) := pR_{E,0}^B(s) + (1-p)R_B^0(s). \] (21)

Note that all these mappings are strictly decreasing. As the notation suggests and shown in Appendix A.1, they determine the returns at the bubbleless steady state with a binding borrowing constraint depending on the savings rate \( s \in \{ s^L, s^H \} \). Lemma A.2 shows that they play a crucial role for existence of bubbly steady states.

5.1 Role of the savings glut

We turn off the savings glut by assuming \( S(x) \equiv s \) for all \( x \in X \). Formally, this can be achieved by choosing \( \rho \in \{0, \infty\} \). Then, by Lemmata A.1 and A.2, the economy has a unique bubbleless steady state \( \bar{x}^0 = (\bar{w}^0, 0) \) and at most one bubbly steady state \( \bar{x} = (\bar{w}, \bar{b}) \), \( \bar{b} > 0 \). The bubbleless steady state lies in \( X_B \) if \( \gamma(s) > 0 \) and in \( X_N \) if \( \gamma(s) \leq 0 \).

There are two generic cases. First, suppose \( R_{E,0}^B(s) < 1 \). In this case, the borrowing constraint will be non-binding at any bubbly steady state (i.e., \( \bar{x} \in X_N \) provided it exists). A restriction necessary and sufficient for \( \bar{x} \) to exist is \( q^0(s) < 1 \), which is precisely the overaccumulation condition in Tirole (1985). In fact, one may view the model as a special case of Tirole (1985) in this first scenario. As the bubbly steady state satisfies \( f'(\bar{k}) = 1 > f'(\bar{k}^0) \), crowding-in is directly excluded.

Second, suppose \( R_{E,0}^B(s) > 1 \). In this case, the borrowing constraint will be binding at the bubbly steady state (i.e., \( \bar{x} \in X_B \) provided it exists), but also at the bubbleless steady state. In this case, a necessary and sufficient condition for \( \bar{x} \) to exist is \( R_B^0(s) < 1 \), i.e., the bubble return must be smaller than unity at the bubbleless steady state. This is precisely the existence condition derived in Kunieda (2008) for an OLG production economy with borrowing constraints. Such a friction induces a spread between the returns on capital and the bubble which makes the existence of a bubbly steady state compatible with initial underaccumulation of capital, i.e., \( f'(\bar{k}^0) > 1 \).

Whether an initially binding borrowing constraint induces a spread between the returns on capital and loans/bubbles or not, it does not affect the resources transferred through the credit market because savers have a low (in fact, zero) intertemporal elasticity of substitution in consumption. Therefore, a bubble necessarily absorbs part of consumer in-
comes which unambiguously leads to crowding-out. This shows why savings have to adjust at the extensive margin for crowding-in to occur.

5.2 Role of the financial friction

The financial friction can be turned off by setting $\lambda = 1$. In this case, by Lemmata A.1 and A.2 the economy has at most one bubbleless steady state $\bar{x}^0 = (\bar{w}^0, 0)$ and at most one bubbly steady state $\bar{x} = (\bar{w}, \bar{b})$, $\bar{b} > 0$. Let us assume that both $\bar{x}^0$ and $\bar{x}$ exist.

By the observations from the previous subsection, crowding-in can not occur if both steady states lie in the same savings regime. Furthermore, a bubble can only induce a switch from low to high savings (i.e., $\bar{x}^0 \in X_L$ and $\bar{x} \in X_H$). If the savings rate is constant, a bubble crowds out investment. But then, returns—equal to the marginal product of capital—would be higher in the bubbly equilibrium which is inconsistent with a decline in savings rates.

Therefore, suppose $\bar{x}^0 \in X_L$ and $\bar{x} \in X_H$, i.e., a savings glut occurs and investment adjusts at the extensive margin. In the absence of the frictions, a savings glut is only compatible with an increase in capital returns which requires a decrease in capital. Thus, the savings glut alone is not sufficient to generate crowding-in. To summarize, the injection of bubbles is capable of triggering the savings glut, but in the absence of financial frictions, this will unambiguously lead to crowding-out.

5.3 Interaction of the financial friction and the savings glut

We saw in the previous subsections why crowding-in requires a switch from low to high savings and a binding borrowing constraint in at least one of the two steady state. Let us now assume $0 < \rho < \infty$ and $\lambda < 1$ and focus on the case $\bar{x}^0 \in X_L$ and $\bar{x} \in X_H$.

Formally, when the borrowing constraint is binding at the bubbly steady state $f'(\bar{k}^0) = \varrho^0(s)$ and $f'(\bar{k}) = \varrho_B(s)$, the borrowing constraint is necessarily binding also at the bubbleless steady state. This and $R^0_B(s) < 1$ imply $\varrho^0(s) < \varrho_B(s)$.

Formally, by Lemma A.1, a bubbleless steady state exists in $X^H = X$ if $\varrho^0(s^H) \geq \rho$, in $X^L = X$ if $\varrho^0(s^L) < \rho$ and fails to exist if $\varrho^0(s^H) < \rho < \varrho^0(s^L)$. By Lemma A.2, a bubbly steady state exists in $X^H$ if $\varrho^0(s^H) < 1$ and $\rho \geq 1$ and in $X^L$ if $\varrho^0(s^L) < 1$ and $\rho > 1$.

Note that by Lemma A.1 $\bar{x}^0 \in X^H$ requires $f'(\bar{k}^0) = \varrho^0(s^H) \geq \rho$ while $\bar{x} \in X^L$ requires $\rho > 1 = f'(\bar{k})$ and $\varrho^0(s^L) < 1$. But then, $\varrho^0(s^H) > \varrho^0(s^L)$ which is impossible since $\varrho^0$ is decreasing.
Why does crowding-in require a binding borrowing constraint at the bubbleless steady state? Suppose the borrowing constraint were non-binding at \( \hat{x}^0 \) (i.e., \( \hat{x}^0 \in X_N^L \)). Then, as demonstrated in the previous section, even if we were to switch to a bubbly steady state \( \hat{x} \) in the high-savings regime where the borrowing constraint is non-binding, crowding-out necessarily occurs due to savings consistency. If, in addition, capital is depressed at the bubbly steady state due to a binding borrowing constraint, this would further amplify and add to the crowding-out effect. Thus, \( \hat{x}^0 \in X_N^L \) is not compatible with an increase in returns necessary to support a savings glut. These insights show why savings must be low and capital investment must be depressed due to a binding borrowing constraint at the bubbleless steady state for crowding-in to occur.

We can now explain the mechanism of crowding-in. In the initial bubbleless steady state, a binding borrowing constraint keeps the return on loans \( \bar{R}^0 \) low relative to the capital return. All capital investment is undertaken by entrepreneurs who earn a high return \( \bar{R}^{E,0} \) while the return on bubbles \( \bar{R}^0 \) is still so low that \( \bar{R}^{S,0} < \rho \) and speculators choose not to save/invest their wealth.

A bubble now offers an alternative investment opportunity to savers. In response they reduce their supply of loans and/or demand a higher return on their credit. This lowers the return spread by increasing the return \( \bar{R} \) on loans and bubbles, and decreasing the return \( \bar{R}^E \) earned by entrepreneurs. At this point, it is crucial that the change in returns increases \( \bar{R}^S \) above the threshold value \( \rho \) to trigger a savings glut. For this to happen, the probability \( p \) must be sufficiently small such that the increase in \( (1 - p)\bar{R} \) overcompensates the reduction in \( p\bar{R}^E \). In particular, it must be that \( p < \frac{1}{2} \). Instead of consuming speculators now save and invest their wealth which adds additional resources to the economy. These additional investments must now overcompensate the resources absorbed by the bubble and add to the formation of capital. This is precisely when crowding-in occurs in our model. It is important to note that crowding-in expands capital investment at the extensive margin. The numerical examples from Section 4.3 show that investment in fact decreases at the intensive margin (\( \bar{i} < \bar{i}^0 \)), i.e., each entrepreneur invests less but the number of entrepreneurs increases.

Finally, recall that out of the group of speculators, only a fraction \( p \) become investors while the remaining \( 1 - p \) become savers. Therefore, the additional formation of capital is not necessarily accompanied by an increase in the resources exchanged through the credit market. In fact, the numerical examples presented above demonstrate that the equilibrium credit volume even decreases in the bubbly equilibrium (\( \bar{l} < \bar{l}^0 \)) making the credit constraint less tight which may even vanish entirely.
5.4 Intensive vs. extensive margin

Motivated by the empirical findings reported in the introduction, our model emphasizes adjustments of savings at the extensive margin to generate crowding-in. The mechanism, however, is also compatible with an adjustment at the intensive margin. To see this, assume that speculators choose their savings to maximize the expected value of a two-period utility function $U(c^y, c^o) = \rho u(c^y) + u(c^o)$, $\rho > 0$ given income $w_t$ and an uncertain return on savings which is $R^E_{t+1}$ with probability $p$ and $R_{t+1}$ with probability $1 - p$. For the specific case $u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}$, $\sigma \geq 0$, the amount of resources saved by each speculator are given by $S_t = s(R^E_{t+1}, R_{t+1}; \sigma)w_t$ where the savings rate is given by

$$s(R^E, R; \sigma) = 1 - \frac{1}{1 + \rho^{-\frac{1}{\sigma}}[p(R^E)^{1-\sigma} + (1 - p)R^{1-\sigma}]}$$

resulting in a smooth aggregate savings function if $\sigma > 0$. Since

$$\lim_{\sigma \to 0} s(R^E_{t+1}, R_{t+1}; \sigma) = \begin{cases} 0 & \text{if } pR^E_{t+1} + (1 - p)R_{t+1} < \rho \\ 1 & \text{otherwise} \end{cases}$$

the savings behavior (3) assumed in our model can be recovered as the limiting case $\sigma = 0$. Moreover, as the function $s$ in (22) depends continuously on $\sigma$, we expect the crowding-in result derived for $\sigma = 0$ to also hold for $\sigma$ sufficiently small in which case capital investment would expand at the intensive margin. We remark, however, that for $\sigma > 0$, the equilibrium conditions become much more involved since the investment decision made by entrepreneurs must be distinguished from those of speculators who become investors. Deriving conditions, under which crowding-in occurs with capital investment expanding at the intensive margin, is beyond the scope of this paper.

6 Dynamics of crowding-in

Starting in an initially bubbleless situation, is there an equilibrium path which converges to a bubbly steady state with higher wage and capital? We show as our main result of this section that the answer is affirmative and crowding-in occurs along the entire equilibrium path towards the bubbly steady state.

To prove this result, we focus on the crowding-in scenario from Section 4.2 with a bubbleless steady state $\bar{x}^0 = (\bar{\bar{w}}, 0) \in X^L_B$ and a bubbly steady state $\bar{x} = (\bar{\bar{w}}, \bar{b}) \in X^H = X^H_N \cup X^H_B$.
where \( \bar{w} > \bar{w}^0 \) and \( \bar{b} > 0 \). We first characterize the local stability properties of either steady state. Denote by \( \Phi^n := \Phi \circ \ldots \circ \Phi, n \geq 0 \) the \( n \)-fold composition setting \( \Phi^0 = \text{id}_X \).

### 6.1 Stability of bubbleless steady states

Let \( \bar{x}^0 = (\bar{w}^0, 0) \in X^L_B \) be a bubbleless steady state in the low-savings regime with a binding borrowing constraint. We would like to characterize the set of bubbleless states \( x_0 = (w_0, 0) \in X^L_B \) attracted by \( \bar{x}^0 \), i.e., which stay in \( X^L_B \) under iteration of \( \Phi \) and converge to \( \bar{x}^0 \). Formally, \( \Phi^n(x_0) \in X^L_B \) for all \( n \geq 0 \) and \( \lim_{n \to \infty} \Phi^n(x_0) = \bar{x}^0 \). Observe from (16a) that the bubbleless dynamics take the one-dimensional form \( w_{t+1} = (1 - \theta)(s_t w_t)^\theta \), \( t \geq 0 \). Hence, for a constant savings rate \( s \), any initial state \( x_0 = (w_0, 0) \) converges to \( \bar{x}^0 \) under iteration of \( \Phi \), even monotonically. Furthermore, a binding borrowing constraint at \( \bar{x}^0 \) requires \( \gamma(s^L) > 0 \) which ensures that it remains binding at any bubbleless state. Thus, it remains to ensure that returns are consistent with a low savings rate along the entire path. Straightforward calculations lead to the following result.

**Lemma 6.1.** A bubbleless steady state \( \bar{x}^0 = (\bar{w}^0, 0) \in X^L_B \) attracts all initial states \( x_0 = (w_0, 0) \) for which \( w_0 > w^0_{\text{crit}} := \bar{w}^0 (R^{S,0}/\rho)^{\frac{1}{1-\theta}} \).

Lemma 6.1 shows that \( \bar{x}^0 \) attracts all bubbleless states where the wage is sufficiently large. Note that this critical value satisfies \( w^0_{\text{crit}} < \bar{w}^0 \) due to \( R^{S,0} < \rho \).

### 6.2 Stability of bubbly steady states

Let \( \bar{x} = (\bar{w}, \bar{b}) \in X^H_j \subset X^H \), \( \bar{b} > 0 \) be a bubbly steady state in the (interior of the) high-savings regime where \( j \in \{B, N\} \) determines whether the borrowing constraint is binding or not. Let \( M \) be the set of points \( x \in X^H_j \) attracted by \( \bar{x} \), i.e., \( \Phi^n(x) \in X^H_j \) for all \( n \geq 0 \) and \( \lim_{n \to \infty} \Phi^n(x) = \bar{x} \). Below, we show that \( \bar{x} \) is saddle-path stable, i.e., the Eigenvalues \( \lambda_1, \lambda_2 \) of the Jacobian matrix \( D\Phi(\bar{x}) \) are real and satisfy \( 0 \leq |\lambda_1| < 1 < |\lambda_2| \). Thus, by the stable-manifold theorem, \( M \) is a one-dimensional subset (manifold) of the state space which is characterized in the next result.

**Lemma 6.2.** Let \( \bar{x} = (\bar{w}, \bar{b}) \in X^H_j \), \( \bar{b} > 0 \), be an interior steady state. Then \( \bar{x} \) is saddle-path stable. Defining \( \bar{m} := s^H(1 - q^0(s^H)/f^\prime(\bar{b})) \) and \( w_{\text{crit}} := \bar{w} (R^S/\rho)^{\frac{1}{1-\theta}} \), \( M \) takes the form:

\[
M = \left\{ (w, b) \in \mathbb{R}^2_+ | w \leq w_{\text{crit}}, b = \bar{m}w \right\}.
\]
Note that the critical value satisfies \( w_{\text{crit}} > \bar{w} \) due to \( \bar{R}_S > \rho \) and that \( \bar{m} > 0 \), as shown in the proof. Geometrically, \( M \) defines an increasing linear curve in the state space which is self-supporting under \( \Phi \), i.e., \( \Phi(M) \subset M \). Thus, any initial state stays in \( M \) under iteration of \( \Phi \). The saddle-path stability of bubbly steady states is a common feature shared by virtually all OLG models with bubbles such as Tirole (1985). In particular, the bubbly equilibrium is determinate in the sense that the initial condition is uniquely determined by the saddle-path \( M \) towards the bubbly steady state.

### 6.3 Multiple equilibria

Supposing the economy starts in an initially bubbleless state \( x_0 = (w_0, 0) \in X^L \), we now ask two questions. First, is it possible to inject a bubble into the system such that long-run investment increases and the economy converges to the bubbly steady state with higher capital? More formally: Is there a value \( b_0 > 0 \) which shifts the initial state to \( x'_0 = (w_0, b_0) \) such that the economy converges to \( \bar{x} \)? Second, which properties do we observe along the equilibrium path starting in \( x'_0 \)? The following main result answers these questions.

**Theorem 6.1.** Let \( \bar{x}^0 = (\bar{w}^0, 0) \in X^L \) and \( \bar{x} = (\bar{w}, \bar{b}) \in X^H, \bar{b} > 0 \) be steady states for which \( \bar{w} > \bar{w}_0 \). Then, we have:

(i) For all \( w_0 < w_{\text{crit}} \) there is a unique \( b_0 > 0 \) such that \( \lim_{t \to \infty} \Phi^t(w_0, b_0) = \bar{x} \).

(ii) If \( w_0 < \bar{w} \), the sequence \( (w_t, b_t) := \Phi^t(w_0, b_0), t \geq 0 \) is strictly increasing.

(iii) If \( \bar{w}^0 \leq w_0 < w_{\text{crit}} \), then \( w_t \geq \bar{w}^0 \) for all times \( t \geq 0 \).

The first assertion employs the explicit form of the stable set \( M \) from Lemma 6.2 and the value \( b_0 \) is such that \( x_0 \in M \). On \( M \), convergence to \( \bar{x} \) is always monotonic. By (ii), if the initial state \( w_0 \) is below \( \bar{w} \), the bubble increases investment immediately and in all future periods relative to \( w_0 \). In the case when \( w_0 = \bar{w} \), the economy is initially in a bubbleless steady state, and crowding-in occurs immediately in \( t = 0 \) and investment continues to increase in all successive periods as the economy converges to the bubbly steady state \( \bar{x} \). Due to the stability properties of \( \bar{x}^0 \) stated in Lemma 6.1, the assumption \( w_0 = \bar{w} \) seems not too restrictive. The findings from Theorem 6.1 are illustrated in Figure 2.

We simulate the evolution of equilibrium variables along the path towards the bubbly steady state for Example 2 studied in Section 4.3. In the initial period \( t = 0 \), the economy is
in a bubbleless steady state $\bar{x}^0 = (\bar{w}^0, 0)$ when a bubble $b_0 > 0$ determined as in Theorem 6.1 is injected into the system. Figure 3 shows the induced time series of the bubble, the wage and the returns as the economy converges to the bubbly steady state $\bar{x} = (\bar{w}, \bar{b})$. The solid lines represent the values at the bubbleless steady state. The dots show the evolution of the variables $(b_t, w_t, R_t, R_t^E)$ in the bubbly equilibrium which converge to $(\bar{b}, \bar{w}, \bar{R}, \bar{R}^E) = (0.18, 0.38, 1.00, 1.03)$.

The injection of a bubble brings the economy on a new equilibrium path along which capital investment, wages, and output are higher and increase in each period $t \geq 0$. The fact that the bubble also grows monotonically along the entire equilibrium path implies $b_{t+1}/b_t = R_t > 1$ for all $t > 0$. This and $\bar{R} > \bar{R}^0$ due to Proposition 4.1 implies that $R_t$ must initially overshoot its long run value and then gradually decline to reach a value permanently higher than in the bubbleless equilibrium. A similar pattern is observed for $R_t^E$ which initially also overshoots before converging to a value permanently lower than in the bubbleless equilibrium as predicted by Proposition 4.1.

Theorem 6.1 determines a critical level for the initial bubble for crowding-in to occur. For initial bubbles close to but lower than the critical level, crowding-in is not persistent and the economy converges back to an asymptotically bubbleless situation (where the adjustment path may alternate between different regimes of the state space). It is also clear from the discussion of the mechanism in Section 5 that for crowding-in (temporary or permanent) to occur at all, the initial bubble must be large enough to trigger a savings glut. The injection of bubbles too small to trigger the savings glut in fact leads to crowding-out of
capital. Qualitatively, this is in line with findings in Martin & Ventura (2010), Hirano & Yanagawa (2010), or Phan & Ikeda (2015).

7 Conclusion

This paper proposes a new mechanism by which bubbles crowd in capital investment. If initial capital investment is depressed by a binding borrowing constraint, a bubbly asset offers an alternative investment opportunity which competes with entrepreneurs for savings. In response, a rise in returns triggers a savings glut by attracting new investors who add additional resources to the economy. When these resources overcompensate the savings absorbed by the bubble, aggregate capital investment expands. This channel operates even when entrepreneurs do not hold bubbles, i.e., in the absence of any collateral effects of bubbles.

The bubble in our economy never bursts. This can easily be modified by assuming as in Weil (1987) or Caballero & Krishnamurthy (2006) that there is a constant exogenous probability for the bubble to burst in the following period. Such a modification could
easily be incorporated and would neither alter the basic mechanism nor any of the main results of this paper. It is, however, not clear whether incorporating random fluctuations of economic fundamentals in our framework gives rise to recurrent stochastic bubbles as defined in Kamihigashi (2011).

In our model, crowding-in is necessarily accompanied by a reduction in the return spread between savers and entrepreneurs. The absence of investment risk does not allow us to relate this observation to the low interest spread usually associated with bubble episodes (c.f. Barlevy, 2013)—the idea that risk is underpriced when bubbles build up. Inclusion of investment risk would therefore be a natural extension of our model. Another interesting extension would be to open up the economy and consider a two country model where global imbalances lower the world interest rate and fuel bubbles. Ikeda & Phan (2015)’s model offers an important benchmark in this direction.

Finally, whether the crowding-in mechanism studied in this paper generalizes to assets with a positive fundamental value such as housing as studied in Hillebrand & Kikuchi (2015) is another issue to be studied in future research.

A Appendix

A.1 Existence of steady states

A.1.1 Bubbleless steady states

Using (20) and (21) the following result provides a complete characterization of the conditions under which bubbleless steady states exist in either of the regimes defined in (19).

Lemma A.1. The map $\Phi$ has a bubbleless steady state $\bar{x}^0 = (\bar{w}^0, 0)$ in

1. $X_N^H$ iff $\gamma(s^H) \leq 0$ and $q^0(s^H) \geq \rho$
2. $X_N^L$ iff $\gamma(s^L) \leq 0$ and $q^0(s^L) < \rho$
3. $X_B^H$ iff $\gamma(s^H) > 0$ and $R_B^S(s^H) \geq \rho$
4. $X_B^L$ iff $\gamma(s^L) > 0$ and $R_B^S(s^L) < \rho$

The second conditions in (i) and (ii) as well as in (iii) and (iv) are mutually exclusive (recall that the functions defined in (21) are decreasing). Thus, in each of the sets $X_N$, $X_B$, $X^H$ and $X^L$ $\Phi$ has at most one bubbly steady state. Further, the bubbleless steady state—if it exists—is unique if either $\gamma(s^H) \leq 0 \land \gamma(s^L) \leq 0$ or $\gamma(s^H) \geq 0 \land \gamma(s^L) \geq 0$. 

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Thus, co-existing bubbleless steady states can only occur if either $\gamma(s^H) < 0 < \gamma(s^L)$ or $\gamma(s^L) < 0 < \gamma(s^H)$, in which case the borrowing constraint is binding in precisely one of them.

To characterize the associated steady state values, denote by $\bar{s}^0 = S(\bar{x}^0)$ the savings rate and $\bar{k}^0 = K(\bar{x}^0, \bar{s}^0)$ the capital stock at the steady state $\bar{x}^0$. In cases (i) and (ii) of Lemma A.1 where the borrowing constraint is non-binding, the steady state returns satisfy

$$\bar{R}^0 = \bar{R}^{E,0} = \bar{R}^{S,0} = f'(\bar{k}^0) = \bar{q}^0(s^0). \tag{A.1}$$

In cases (iii) and (iv) where the borrowing constraint is binding, they are

$$\bar{R}^0 = R_B^0(s^0) < f'(\bar{k}^0) = \bar{q}^0(s^0) < \bar{R}^{E,0} = R_B^{E,0}(s^0), \bar{R}^{S,0} = p\bar{R}^{E,0} + (1-p)\bar{R}^0. \tag{A.2}$$

Note that $f'(\bar{k}^0) = \bar{q}^0(s^0)$ independently of whether the borrowing constraint is binding or not. Thus, the same holds for the steady state wage which can be written as

$$\bar{w}^0 = (1-\theta)\theta^{\frac{\rho}{1-\rho}} f'(\bar{k}^0)^{-\frac{\rho}{1-\rho}}. \tag{A.3}$$

A.1.2 Bubbly steady states

The following result offers a complete characterization of the conditions under which bubbly steady states exist in each of the four regimes defined in (19).

**Lemma A.2.** The map $\Phi$ has a bubbly steady state $\bar{x} = (\bar{w}, \bar{b})$, $\bar{b} > 0$ in

(i) $X_N^H$, iff $q^0(s^H) < 1$, $R_B^{E,0}(s^H) < 1$, and $\rho \leq 1$.

(ii) $X_N^L$, iff $q^0(s^L) < 1$, $R_B^{E,0}(s^L) < 1$, and $\rho > 1$.

(iii) $X_B^H$, iff $R_B^0(s^H) < 1 < R_B^{E,0}(s^H)$, and $1 - p + pR_B^{E,0}(s^H) \geq \rho$.

(iv) $X_B^L$, iff $R_B^0(s^L) < 1 < R_B^{E,0}(s^L)$, and $1 - p + pR_B^{E,0}(s^L) < \rho$.

The last conditions (involving $\rho$) in cases (i) and (ii) as well as in (iii) and (iv) are again mutually exclusive. The same is true of the requirements in (i) and (iii) as well as in (ii) and (iv). Thus, in each of the sets $X_N, X_B, X^H$ and $X^L$, $\Phi$ has at most one bubbly steady state. Of course, the existence conditions do not require a bubbleless steady state to exist in the same regime.
To characterize the associated steady state values, denote by $\bar{s} = S(\bar{x})$ the savings rate and by $\bar{k} = K(\bar{x}; \bar{s})$ the capital stock associated with $\bar{x}$. Then, the steady state returns in cases (i) and (ii) satisfy

$$\bar{R} = R^S = R^E = f'(\bar{k}) = 1.$$  \hspace{1cm} (A.4)

In cases (iii) and (iv) they are given by

$$\bar{R} = 1 < f'(\bar{k}) = \varrho_B(\bar{s}) < \bar{R}^E = R^E_B(\bar{s}) \quad \text{and} \quad \bar{R}^S = p\bar{R}^E + 1 - p.$$  \hspace{1cm} (A.5)

Using the previous definitions the steady state values can be expressed as$^{12}$

$$\bar{w} = (1 - \theta)\bar{\vartheta} \frac{\varrho}{f'(\bar{k})}, \quad \bar{b} = \bar{s}\bar{w} \left(1 - \frac{\varrho(\bar{s})}{f'(\bar{k})}\right).$$  \hspace{1cm} (A.6)

### A.2 Parameter conditions for crowding-in

The following theorem complements Theorems 4.1 and 4.2 by providing a complete characterization of the classes of economies for which crowding-in occurs.

**Theorem A.1.** The classes $\mathcal{E}_1$ and $\mathcal{E}_2$ take the following explicit form:

(i) $\mathcal{E}_1 = \left\{(a, \rho, \lambda, \varrho, \theta) \in \mathcal{E} \mid \lambda < \frac{1}{2}, \theta'\frac{\pi(a; p)}{a} < \rho \leq 1, \theta'\frac{1 - \lambda}{\pi(a; p)} < 1, 2a < \theta' < 1 \right\}$

(ii) $\mathcal{E}_2 = \left\{(a, \rho, \lambda, \varrho, \theta) \in \mathcal{E} \mid \lambda < \frac{1}{2}, \theta'\frac{\pi(a; p)}{a} < \rho \leq 1 - p + p\theta'\frac{1 - \lambda}{\pi(a; p)}, 2a < \rho + \lambda\theta' < \min\{1, \theta'\} \right\}$

where $\pi(x; p) := x(1 - p) + p(1 - x)$ and $\theta' := \frac{\theta}{1 - \sigma}$.

To develop some intuition for the restrictions in Theorem A.1, recall that each member economy of the classes $\mathcal{E}_1$ and $\mathcal{E}_2$ has (a) a bubbleless steady $\bar{x}^0$ in $X_B^L$, (b) a bubbly steady state $\bar{x}$ in the high-savings regime $X^H = X_N^H \cup X_B^H$, (c) capital stocks which satisfy the crowding-in condition $f'(\bar{k}) > f'(\bar{k})$. We will link the restrictions defining the classes

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$^{12}$To see that the second quantity in (A.6) is positive, i.e., $\frac{\varrho(\bar{s})}{f'(\bar{k})} < 1$ note that $\bar{x} \in X_B$ implies $f'(\bar{k}) = 1$ and $\varrho(\bar{s}) < 1$ by Lemma A.2(i),(ii) while $\bar{x} \in X_N$ requires $R^E_B(\bar{s}) < 1$ due to Lemma A.2(iii),(iv) which again implies $\frac{\varrho(\bar{s})}{f'(\bar{k})} = \frac{\varrho(\bar{s})}{\varrho_B(\bar{s})} < 1$. Thus, indeed $\bar{b} > 0$. The economic reason is a crowding out effect that occurs between the bubbleless and bubbly steady states that lie in the same savings regime.
in Theorem A.1 to each of these three requirements. In this regard, the parameter values 
\((a, p, \rho)\) essentially govern the savings glut in the model economy, \(\lambda\) controls the friction, 
and \(\theta'\) represents the capital return induced by the production technology which plays a 
key role in the existence conditions for steady states.

(a) \(\bar{x}^0 \in X^L_B\). First, observe that a binding borrowing constraint at \(\bar{x}^0\) requires the financial friction to be sufficiently tight such that \(\lambda < \frac{1}{2}\). The intuition is straightforward: At a bubbleless state in the low-savings regime, only pure entrepreneurs and pure savers interact through the credit market. Therefore, half of the investment is financed by the entrepreneurs’ own income while the other half is financed by credit. As the capital return exceeds the return on loans, the borrowing constraint can never bind as long as an entrepreneur can credibly pledge at least half of her project revenue, i.e., \(\lambda \geq \frac{1}{2}\). Therefore, the restriction \(\lambda < \frac{1}{2}\) is necessary for the borrowing constraint to bind.\(^{13}\)

Second, the reservation return must be sufficiently large relative to the (shadow) return of speculators to ensure that the bubbleless steady state is in the low savings regime. This requirement is embodied in the condition \(\pi(\lambda; p) \frac{\theta'}{a} < \rho\). Here \(\theta'/a\) represents the return on capital adjusted by the population share of savers \(a\) in the low-savings regime. If the borrowing constraint were absent, this would be the return earned by all consumers. A binding borrowing constraint now induces a return spread due to which entrepreneurs earn a higher return equal to \((1 - \lambda) \frac{\theta'}{a}\) and savers a lower return equal to \(\lambda \frac{\theta'}{a}\). Clearly, this spread decreases as \(\lambda\) increases and vanishes gradually as \(\lambda\) approaches \(\frac{1}{2}\). Further, as \(p \geq \frac{1}{2}\) would imply \(\frac{\theta'}{a} \pi(\lambda; p) \geq \frac{1}{2} \frac{\theta'}{a} > 1 \geq \rho\) by the other parameter restrictions, we also infer that \(p < \frac{1}{2}\) is required (the economic reason for this is explained in Section 5).

From these results, conclude that the return \(\theta'/a\) is strictly increasing in \(p\) and \(\lambda\). The non-linear curve \(C_1 := \{(p, \lambda) \in [0, 1/2]^2 | \pi(\lambda; p) = \rho a/\theta'\}\) defines the upper boundary of the projection in Figure 1.\(^{14}\)

(b) \(\tilde{x} \in X^H\). First, as stated in Lemma A.2 the existence of a bubbly steady in the high savings regime requires a bubble return smaller than unity supporting the bubbleless steady state in this regime. These existence conditions are further discussed in Section 5. If \(\tilde{x} \in X^H_{N^*}\), this condition simply reads \(\theta' < 1\) which is precisely the overaccumulation condition in Tirole (1985). If \(\tilde{x} \in X^H_B\), the condition reads \(\theta'/(1 - \pi(a; p)) < \frac{1}{\lambda}\). Similar to (a), the l.h.s may be interpreted as a capital return adjusted by the population share of

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\(^{13}\)Clearly, a critical value different from \(\frac{1}{2}\) would emerge if we deviated from the assumption that the population share of pure savers and entrepreneurs is identical.

\(^{14}\)Note that \(C_1\) is the graph of a smooth function which is strictly decreasing and strictly concave if \(2a < \theta'/\rho\) which holds in the example depicted in Figure 1.
savers $1 - \pi(a; p)$ in the high-savings regime. This return must be small relative to the tightness of the friction which is measured by $\lambda^{-1}$. As in similar models in the literature, the presence of a friction permits to relax the existence condition. The second condition defines a linear curve $C_2 := \{(p, \lambda) \in [0, 1/2]^2 | \lambda \theta' = 1 - \pi(a; p)\}$ in $p$-$\lambda$ space which is an upper bound of the projection of $E_2$ in Figure 1. Second, due to Lemma A.2 the sign of $1 - \theta' \frac{1 - \lambda}{\pi(a; p)}$ essentially determines whether or not the borrowing constraint is binding at the bubbly steady state, i.e., whether $\bar{x} \in X_H^B$ or $\bar{x} \in X_N^H$. This seems in line with common intuition that the borrowing constraint becomes non-binding if $\lambda$ is sufficiently large relative to the other parameter values. The set $C_3 := \{(p, \lambda) \in [0, 1/2]^2 | \lambda \theta' = \theta' - \pi(a; p)\}$ defines an affine-linear downward-sloping line in $p$-$\lambda$ space which separates the two sets $E_1$ and $E_2$ in Figure 1. Note that a value $\theta' < 1$ in the example ensures that the curve $C_2$ defined above always lies above $C_3$, i.e., the restriction represented by $C_2$ is automatically satisfied for points which lie below $C_3$. Finally, the reservation return must be sufficiently small for the bubbly steady state to be in the high savings regime. In the case of a non-binding borrowing constraint, this simply requires $\rho \leq 1$ while $\rho \leq 1 - p + p\theta' \frac{1 - \lambda}{\pi(a; p)}$ must hold if the borrowing constraint is binding at the bubbly steady state. Specifically, if $\rho \leq 1$ which holds in the example in Figure 1, the second condition is automatically satisfied due to $\theta' \frac{1 - \lambda}{\pi(a; p)} > 1$. Also note that there is a tension between this restriction on $\rho$ and the one discussed in (a).

(c) $f'(\bar{k}) > f'(\bar{k})$. If $\bar{x} \in X_H^H$, the condition reads $2a < \theta'$ while $2a < \lambda \theta' + \pi(a; p)$ must hold in cases where $\bar{x} \in X_H^B$. Note that both conditions require the population share $a$ to be small relative to the capital return $\theta'$. As the savings glut raises the savings rate from $s^L = 2a$ to $s^H = 1$, the restriction ensures that this effect is sufficiently strong, which seems quite intuitive for crowding-in to occur. Again, there is a tension between this condition and the existence condition discussed (first) in (b). The second condition defines a downward-sloping affine-linear curve $C_4 := \{(p, \lambda) \in [0, 1/2]^2 | \lambda \theta' = a - p(1 - 2a)\}$ in $p$-$\lambda$ space which is the lower boundary of the set $E_2$ in Figure 1.

Summarizing, we see that the curves $C_1$ and $C_3$ define the lower and upper boundaries of the (projected) class $E_1$ while the lower and upper boundaries of $E_2$ are defined by $C_4$ and the minimum of the curves $C_1$, $C_2$, and $C_3$. These restrictions reflect the basic tension between the mechanisms of the model under which crowding-in occurs and the conditions under which steady states in the respective regimes exist.
A.3 Proofs

Proof of Lemma A.1. As the question whether or not the borrowing constraint is binding depends exclusively on the ratio \( \frac{\bar{s}}{\bar{w}} \) (which is zero at any bubbleless steady state) relative to \( \gamma(s) \), the sign of the latter determines the regime in which the bubbleless steady state lies. One can then show by direct computations using equations (14b)–(14e) that the functions defined in (21) determine the steady state returns in the respective regime. The second condition ensures consistency with the behavior of speculators. □

Proof of Lemma A.2. Proof of Lemma A.2. Define \( m_t := \frac{b_t}{w_t} \). Then, using (16a,b) and the Cobb-Douglas specification, one obtains the following relation that holds for each \( t \geq 0 \):

\[
m_{t+1} = \phi(m_t; s_t) := \frac{\theta}{1 - \theta s_t - \alpha(s_t) - m_t} \min \left\{ \lambda, \frac{s_t - \alpha(s_t) - m_t}{s_t - m_t} \right\}.
\]  \tag{A.7}

As the savings rate \( s_t = S(x_t) \) can not be written as a function of \( m_t \), (A.7) does not directly define a dynamical system in \( m \). Observe, however, that for any bubbly steady state \( \bar{x} = (\bar{w}, \bar{b}) \) of \( \Phi \) with steady state savings rate \( \bar{s} = S(\bar{x}) \), the ratio \( \bar{m} := \frac{\bar{b}}{\bar{w}} \) must be a steady state of \( \phi(\cdot; \bar{s}) \), i.e., \( \bar{m} = \phi(\bar{m}; \bar{s}) \) and \( 0 < \bar{m} < \bar{s} \). In addition, the steady state returns must be consistent with the behavior of speculators. Evaluating these conditions separately for each of the four regimes gives the conditions of the lemma. □

Proof of Theorem A.1. Non-emptiness is a direct consequence of the examples. The conditions stated can be verified directly by solving the parameter restrictions using the results from Lemmata A.1 and A.2 and Theorem 4.1 together with equations (14b), (18), (20) and (21). □

Proof of Lemma 4.1. The proof exploits that \( \bar{w} > \bar{w}^0 \) iff \( f'(\bar{k}) < f'(\bar{0}) \) due to (A.3), (A.6).

(i) Assume by contradiction that both \( \bar{x}^0 \) and \( \bar{x} \) lie in \( X^H \) (the proof for \( X^L \) is analogous). Suppose \( \bar{x} \in X^H_{B^0} \). Then, \( f'(\bar{k}) < f'(\bar{0}) \) iff \( 1 < q_0^0(s^H) \), contradicting Lemma A.2(i). Suppose \( \bar{x} \in X^H_B \). Then, \( f'(\bar{k}) < f'(\bar{0}) \) iff \( q_B^H(s^H) < q^0(s^H) \), which can be rearranged to \( R_B^0(s^H) > 1 \), contradicting Lemma A.2(iii).

(ii) By contradiction, let \( \bar{x}^0 \in X_N \). Then, by (i), \( \bar{x}^0 \in X^H_{N^0} \) and \( f'(\bar{0}) = q^0(s^L) \) due to (A.1). First, suppose \( \bar{x} \in X^H_{N^0} \). Then, \( f'(\bar{k}) < f'(\bar{0}) \) iff \( q_0^0(s^L) > 1 \). Savings consistency requires \( f'(\bar{0}) = q^0(s^L) < \rho \leq 1 = f'(\bar{k}) \) by Lemmata A.1(ii) and A.2(i), which is a contradiction. Second, suppose \( \bar{x} \in X_B \). Then, by (A.1) and (A.5), \( f'(\bar{0}) > f'(\bar{k}) \) iff

\[
q^0(s^L) > q_B^H(s^H). \tag{A.8}
\]
The savings consistency conditions from Lemmata A.1(ii) and A.2(iii) yield

\[
q^0(s^L) < \rho < 1 - p + pR_{E,0}^B(s^H).
\] (A.9)

We also know from (A.5) that \(q_B(s^H) > 1\). Thus, \(q^0(s^L) > 1\) by (A.8) and, therefore, \(1 - p + p^0(s^L) < q^0(s^L)\). Using this last result in (A.9) implies \(q^0(s^L) < R_{E,0}^B(s^H)\). By Lemma A.1(ii) \(\gamma(s^L) \leq 0\) which implies \(R_{E,0}^B(s^L) \leq q^0(s^L)\) by (21). Combining both results gives \(R_{E,0}^B(s^L) \leq q^0(s^L) < R_{E,0}^B(s^H)\) which is impossible, since \(R_{E,0}^B\) defined in (21) is decreasing in \(s\) proving the claim.

**Proof of Theorem 4.1.** By (A.3) and (A.6), \(\vec{w} > \tilde{w}^0\) iff \(f'(\bar{k}) < f'(\bar{r}^0)\). Using (A.2) and (A.5) gives precisely the conditions depending on whether \(\bar{x} \in X^H_N\) or \(\bar{x} \in X^H_B\).

**Proof of Theorem 4.2.** Non-emptiness is a direct consequence of the examples in Section 4.3. Restricting the conditions in Theorem A.1 to strict inequalities and using the continuity of \(\pi\) defines an open subset of parameters for which crowding-in occurs.

**Proof of Proposition 4.1.** When bubbles crowd in investment, \(\bar{R}_{E,0} > \bar{R} > \bar{R}_{E}^E > 0\) and \(\bar{R}_{E}^E > \bar{R}^E\). From Lemma 3(i), crowding-in requires a savings glut. Hence, \(\bar{R}_{E,0} > \bar{R}_{E}^S\) which is equivalent to \((1 - p)(\bar{R} - \bar{R}^0) > p(\bar{R}_{E,0}^E - \bar{R}^E)\). As \(\bar{R}_{E,0}^E > \bar{R}^E\) by the first part of this proof, \(\bar{R} - \bar{R}^0 > 0\).

**Proof of Lemma 6.2.** Evaluating the trace and determinant of the Jacobian \(D\Phi(\bar{x})\) gives \(\det D\Phi(\bar{x}) > 0\) and \(\text{tr}D\Phi(\bar{x}) > 1 + \det D\Phi(\bar{x}) > 0\) which implies saddle-path stability.

To construct the representation of \(\text{M}\), let \(X_0 = (w_0, b_0) \in \text{M}\) be arbitrary and \(x_t := \Phi^t(x_0), t \geq 0\). As \(\Phi(\text{M}) \subset \text{M}, x_t \in \text{M}\) for all \(t \geq 0\). Further, \(\text{M} \subset X^H_t\) implies \(s_t = S(x_t) \equiv s^H\) for all \(t\) and the borrowing constraint is either always \((j = B)\) or never \((j = N)\) binding. Thus, the sequence \(m_t := \frac{b_t}{w_t}, t \geq 0\) is generated by the map \(\phi(\cdot; s^H)\) from (A.7) and satisfies \(m_t < \gamma(s^H)\) if \(j = B\) and \(\gamma(s^H) \leq m_t < s^H\) if \(j = N\) for all \(t \geq 0\). As \(\lim_{t \to \infty} (w_t, b_t) = (\bar{w}, \bar{b})\) by definition of \(\text{M}\), \(\lim_{t \to \infty} m_t = \bar{m} := \frac{\bar{b}}{\bar{w}}\) where \(0 < \bar{m} < s^H\).

For both \(j \in \{B, N\}, (m_t), t \geq 0\) is generated by a map of the form \(\tilde{\phi}(m) = \frac{a_0}{a_1 - m}, 0 < m < a_1\) where \(a_0 = \lambda_{t^0}^{\theta} - a_1 s^H - a(s^H)\) if \(j = B\) while \(a_0 = \theta_{t^0} - a_1 s^H\) if \(j = N\). One verifies directly that \(\tilde{\phi}\) has precisely two steady states \(\bar{m}_0 = 0\) and \(\bar{m} = a_1 - a_0\). By Lemma A.2, \(\bar{m} > 0\) in both cases as \(R_{E,0}^B(s^H) < 1\) if \(j = B\) and \(q^0(s^H) < 1\) if \(j = N\). As \(\tilde{\phi}'(\bar{m}) = 1 + \bar{m}/a_0 > 1\), \(\bar{m}\) is unstable while \(\bar{m}_0\) is asymptotically stable.
By this observation, we claim that \((w_0, b_0)\) must satisfy \(m_0 = \frac{b_0}{w_0} = \bar{m}\). By contradiction, suppose \(m_0 < \bar{m}\). Then, stability of \(\bar{m}_0\) implies \(\lim_{t \to \infty} m_t = \bar{m}_0 = 0\) which contradicts \(\lim_{t \to \infty} m_t = \bar{m}\). Conversely, suppose \(m_0 > \bar{m}\). Then, as \(\lim_{m \to a_1} \phi(m) = \infty\), the sequence \(\{m_t\}_{t \geq 0}\) would grow without bound such that \(m_t > a_1\) after finitely many periods, which violates \(m_t < \gamma(s^H) < a_1\) if \(j = B\) and \(m_t < s^H\) if \(j = N\). Conclude that indeed \(\frac{b_0}{w_0} = \bar{m}\) which implies \(\frac{b_t}{w_t} = \bar{m}\) for all \(t \geq 0\) and \(M\) is a subset of \(\{(w, b) \in \mathbb{R}^2_+ \mid b = \bar{m}w\}\). Also note that (A.4) and (A.5) permit \(\bar{m}\) to be written as in the lemma.

Finally, \(b_t = \bar{m}w_t\) and \(s_t \equiv s^H\) permit the return (14e) earned by speculators to be written as \(R_{t+1}^S = \frac{f'(k_t+1)}{f'(k)} R^S = (\frac{w_t}{w})^{\theta-1} \bar{R}^S\) for each \(t \geq 0\). As the sequence \(\{w_t\}_{t \geq 0}\) converges monotonically to \(\bar{w}\), the additional conditions of the lemma restrict the initial value \(w_0\) such that \(R_{t+1}^S \geq \rho\) for all \(t\).

**Proof of Theorem 6.1.** The proof follows directly from Lemma 6.2 and the form of the stable set \(M\). Recall from the definition of \(M\) that \(\Phi(M) \subset M\) such that \((w_t, b_t) \in M\) and \(b_t = \bar{m}w_t\) for all \(t \geq 0\). In particular, \(w_{t+1} = (1 - \theta)(s^H - \bar{m})^\theta w_t^\theta\) for all \(t \geq 0\) which converges monotonically to \(\bar{w}\).

**References**


